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THE APPLICATION OF FINITE ELEMENT METHOD IN CALCULATION OF THIN SHELL J INTEGRAL

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INTRODUCTION

The analysis of cracked structures is a more or less standard procedure within the theory of elasticity. Also, solutions for many elastic-plastic problems as well as for other relevant fracture mechanics areas are available over the recent years (e.g. fatigue, impact or creep fracture). Numerous handbooks for stress intensity factors' calculation are the best proof of what is said above. However, even within linear elastic fracture mechanics (LEFM), it is impossible to obtain adequate expressions for stress intensity factors using simple analogy with 2D plane state analysis of thin shells. Essentially, the problem lies within the fact that the thin shell is a curved three-dimensional (3D) object, which makes its analysis impossible in 2D. The relation between membrane and shear stresses, obviously existing as well as the significant effect of transversal shear stresses make questionable the analyses where these effects are neglected /1,2,3/. Hence, the second fracture mechanics parameter, the J integral, is used, which is within the two-dimensional linear (plane) elasticity reduced to stress intensity factor (SIF), i.e. the crack growth force or the energy release rate, but which is applicable to non-linear elasticity and can be derived as a consequence of the conservation law.

The discovery and application of J integral /4,5,6/ has opened new possibilities in applying fracture mechanics in engineering as well as in its following theoretical development. The most important properties of the J integral are:

It is independent of integration path which allowed simple analytical, numerical and experimental determination.

Its physical interpretation of crack growth force, i.e. energy release rate per unit crack length.

The interpretation of parameters characteristic for the stress state around the crack tip, not only within linear, but also within non-linear elasticity.

Interpretation of conservation law consequences, i.e. the translation invariants of corresponding functionals.

These properties, in a certain way, make the J integral a general fracture mechanics parameter. Its generality, however, is bounded by limits in which the above mentioned properties exist, and these are: non-linear two-dimensional (plane) elasticity, stationary and through crack, static loading and homogenous isotropic material. Considered here is a linear elastic thin shell made of homogenous isotropic material, under static loading, with a through stationary crack. In other words, the possibility of applying the J integral parameter out of its range of planar problems is being considered. The problems which occur result from the fact that there is no general thin shell theory, not even within linear elasticity. The need to reduce a three-dimensional to a two-dimensional problem, presented in almost every thin shell theory, is justified and understandable. However, the criteria for certain neglecting must be taken into account. Neglecting of shell curvature and/or the relation between membrane and shear stresses, as well as transversal shear, cannot be accepted in cracked thin shell analysis. All papers on thin shell J intgeral published so far have one of the two key disadvantages: they either consider the problem in Cartesian orthogonal coordinates /7/, or consider a special form of middle shell surface /8,9/, usually circular cylindrical. Paper /10/ is an exception, but the expressions derived in it are not given in adequate form for further application.

Considering everything here said, an expression of thin shell J integral will be derived taking into account cited the J integral properties, and it will be based on the general thin shell theory, which will be considered in detail in the following text.

THIN SHELL DEFINITION AND INTRODUCTORY COMMENT

In order to describe (define) a shell-shaped body, adequate coordinate systems are introduced: fixed Cartesian coordinate system y^i (i = 1,2,3) and the convective curvilinear coordinate system θ^a $\{\theta^\alpha,\xi\}$ (a = 1,2,3) for which the following transformations are valid:

$y^i = y^i\left(\theta^a\right), \theta^a = \theta^a\left(y^i\right)$ with the condition $\det \frac{\partial y^i}{\partial \theta^a} \neq 0$

The indices (a,b,i,j...) designate three-dimensional quantities, and greek symbols (α,β,δ,...) designate two-dimensional ones, as is usual in literature. For shell-shaped body description convenience (especially in case of thin shells), the convective coordinate system is "separated" into θ^α and $\theta^3 \equiv \xi$, therefore $\theta^a = \{\theta^\alpha,\xi\}$.

The term shell refers to a body that consists of a surface in an Euclid three-dimensional space and a deformable directed vector (called the director) assigned to every surface particle. The surface particles are identified using convective coordinates θ^α, as for the referent and deformed surface configuration, symbols ζ and σ are introduced, respectively. There is also the radius vector $x_i = x_i(\theta^\alpha,t)$, relative to the fixed origin. This vector determines the particle θ^α position in a deformed configuration at instant t. It is now possible to define surface base vectors as partial derivatives of the radius vector using convective coordinates:

$$\underset{\sim}{a}_\alpha = \frac{\partial x^i}{\partial \theta^\alpha} = x^i_\alpha \,, \quad \underset{\sim}{a}_3 = \frac{\partial x^i}{\partial \xi} = x^i_3$$

The oriented thickness vector $h^i = h^i(\theta^\alpha,t)$, which has the purpose of being the director in the Kosera theory /12/, is also introduced. The vector h^i does not need to be perpendicular to the surface σ, which allows to consider transversal shear. Also, this vector is introduced as the generalization of the director $d^i = d^i(\theta^\alpha,t)$ (see /13/ for $h^i = hd^i$, where h represents shell thickness, allowing to consider shells with variable thickness (but it is not the point here). It is to be mentioned that the symbols of quantities introduced above, in case of a referent configuration for which the initial (t = 0) undeformed configuration is taken: Y^i is the radius vector, H^i is the director which is in that case perpendicular to the surface ζ, $A_{\sim\alpha}$ and $A_{\sim 3}$ are the base vectors.

The continuum defined in such way is based on the so-called Kosera surface which is not only a 2D surface, for it contains the director as well. The director actually describes the "thickness" around the surface $\xi = 0$, hence its component along the surface perpendicular can be considered as a representative of the 3D shell thickness.

Finally, one can say that thin shells represent a body for which is valid $H/L \ll 1$ and $H/R \ll 1$, where H, L and R are the thickness, characteristic length and shell radius, in respect.

BASIC CINEMATIC RELATIONS AND METRIC TENSORS

It is assumed that the radius vector $y^i(\theta^a,t)$ is the analytical function of coordinate ξ in the range of $h_1 < \xi < h_2$, and that $\xi = h_1$ and $\xi = h_2$ represent equations for the shell upper and lower limit surfaces. Based on this assumption, y^i can be expanded in a Taylor series:

$$y^i = x^i + \sum_{n=1}^{\infty} \xi^n \frac{1}{n!}\frac{\partial^n y^i}{\partial \xi^n} \tag{1}$$

where x^i is the radius vector of the middle surface particle ($\xi = 0$), given in Cartesian coordinates. By introducing the director expression $d^i_n = \frac{1}{n!}\frac{\partial^n y^i}{\partial \xi^n}$, it follows:

$$y^i = x^i + \sum_{n=1}^{\infty} \xi^n d^i_n \tag{1a}$$

Since thin shells are being considered here, small quantities of second order will be neglected which produces, after the introduction of the symbol $d^i = d_1{}^i$:

$$y^i = x^i + \xi d^i \tag{2}$$

Now the non-dimensional coordinate $\zeta = \frac{2}{h}\xi$ can be introduced, from which it follows:

$$y^i = x^i + \frac{h}{2}\zeta d^i = x^i + \frac{1}{2}\zeta h^i \tag{2a}$$

where $h^i = hd^i$ is the relation between the "classic" director /13/ and the director used here. By differentiating Eq. [2a] through time it follows:

$$\dot{y}^i = \dot{x}^i + \frac{1}{2}\zeta \dot{h}^i = \left[\dot{x}^i \dot{\phi}^i\right]\left\{-\frac{\zeta}{2} e^1_{ijk} h^j\right\} \tag{3}$$

where $\dot{h}^i = -e_{ijk}\dot{\phi}^i h^j$ is expressed using the ϕ^k rotation, which appeared to be necessary in thin shell analysis using finite elements /14/. The assumption from Eq. [3] that the thickness does not change in time (h = 0) is also valid.

In a referent configuration analogous expressions are valid with the exception that it is always possible to choose θ^a in such a way that $H_n{}^i = 0$ for $n > 2$, the expression:

$$Y^i = X^i + \frac{1}{2}\zeta H^i \tag{4}$$

is correct and not an approximation like Eq. [2].

The movement vector v^i can be now defined:

$$v^i = y^i - Y^i = x^i - X^i + \frac{1}{2}\zeta\left(h^i - H^i\right) = u^i + \frac{1}{2}\zeta\kappa^i \tag{5}$$

and from this follows the so called mean configuration:

$$\bar{y}^i = y^i - \frac{1}{2}v^i = Y^i + \frac{1}{2}v^i = \frac{1}{2}\left(y^i + Y^i\right) \tag{6}$$

The expression for the fundamental metric tensor is:

$$g_{\alpha\beta} = y^i_\alpha y^i_\beta = x^i_\alpha x^i_\beta + \frac{1}{2}\zeta\left(x^i_\alpha h^i_\beta + x^i_\beta h^i_\alpha\right) + \frac{1}{4}\zeta^2 h^i_\alpha h^i_\beta \tag{7}$$

having in mind that $y^i_\alpha = x^i_\alpha + \frac{1}{2}\zeta h^i_\alpha$. According to the criterion of neglecting small quantities of the second order and the assessment of $x^i_\alpha = 0(L)$ and $h^i_\alpha = 0(H)$, it is:

$$g_{\alpha\beta} = x^i_\alpha x^i_\beta + \frac{1}{2}\zeta\left(x^i_\alpha h^i_\beta + x^i_\beta h^i_\alpha\right) \tag{8}$$

Similarly, the remaining metric tensor components can be defined:

$$g_{\alpha 3} = y^i_\alpha y^i_3 = \frac{1}{2}x^i_\alpha h^i + \frac{1}{4}\zeta h^i_\alpha h^i = \frac{1}{2}x^i_\alpha h^i \tag{9}$$

because $y_3^i = \frac{h^i}{2}$ and $g_{33} = y_3^i y_3^i = \left(\frac{h}{2}\right)^2 = 0$ (10)

In a referent configuration $H^i X^i{}_\alpha = 0$ and $H^i H^i{}_\alpha = 0$, because of the fact that H^i is perpendicular to the middle surface and that $X^i{}_\alpha$ and $H^i{}_\alpha$ are tangential, therefore:

$$G_{\alpha\beta} = X_\alpha^i X_\beta^i + \frac{1}{2}\left(X_\alpha^i H_\beta^i + X_\beta^i H_\alpha^i\right),\ G_{\alpha 3} = 0,\ G_{33} = 0 \quad (11)$$

Finally, other expressions are introduced for next use:

$$a_{\alpha\beta} = x_\alpha^i x_\beta^i,\ A_{\alpha\beta} = X_\alpha^i X_\beta^i,\ \lambda_{\alpha\beta} = x_\alpha^i h_\beta^i,\ \Lambda_{\alpha\beta} = X_\alpha^i H_\beta^i$$

THE RELATION BETWEEN STRAIN AND DISPLACEMENT

It is necessary to define first the stress tensor in adequate coordinates. This is the o_{ij} pseudotensor σ^{ab} which is obtained by transforming Cauchy's stress tensor t^{ij} from Cartesian coordinates to convective curvilinear coordinates:

$$\sigma^{ab} = \frac{\partial\theta^a}{\partial y^i} \cdot \frac{\partial\theta^b}{\partial y^j} t^{ij} \quad (12)$$

Apart from σ^{ab}, the S^{ab} tensor will also be used, which is obtained by reducing σ^{ab} to the referent configuration:

$$S^{ab} = \sqrt{\frac{g}{G}}\sigma^{ab} = \frac{\rho}{\rho_0}\sigma^{ab} \quad (13)$$

where g and G represent the metric tensor determinants, while ρ and ρ_o represent densities in current and referent configuration, respectively.

The relation between strain and displacement for thin shells is developed using Galerkin's procedure for the following expression:

$$\int \sigma^{ab}\left(\gamma_{ab}^i - \bar{y}_a^i v_b^i\right) dv = 0 \quad (14)$$

which is valid, because of the following definition:

$$\gamma'_{ab} = \bar{y}_a^i v_b^i \quad (15)$$

This strain measure is introduced because of [14], which can produce an incorrect result if the common symmetric tensor has been used in it: $\gamma_{ab} = \frac{1}{2}\left(\bar{y}_a^i v_b^i + y_b^i v_a^i\right)$.

One should have in mind that introducing γ' is possible because of tensor S^{ab} symmetry, hence:

$$S^{ab}\gamma_{ab} = S^{ab}\gamma_{(ab)} = S^{ab}\gamma'_{(ab)} = S^{ab}\gamma'_{ab}.$$

Galerkin's procedure is applied to [14], which was previously "transformed" into a referent configuration:

$$\int S^{ab}\left(\gamma'_{ab} - \bar{y}_a^i v_a^i\right) dV = 0$$

where $dV = (\rho/\rho_o)dv$, which can be written as:

$$\int\left[S^{\alpha\beta}\left(\gamma'_{\alpha\beta} - \bar{y}_\alpha^i v_\beta^i\right) + S^{\alpha 3}\left(\gamma'_{\alpha 3} - \bar{y}_\alpha^i v_3^i\right) + S^{3\alpha}\left(\gamma'_{3\alpha} - \bar{y}_3^i v_\alpha^i\right)\right] dv = 0 \quad (16)$$

where tensor S^{ab} is "separated" by indices 1 and 2 in central shell plane and 3 in the direction of thickness. For further transformation of Eq. [16], it is necessary to express tensor S^{ab} using common stress tensors in thin shell theory: the membrane stress $N^{\alpha\beta}$, the bending stress $M^{\alpha\beta}$ and the transversal shear V^α. It is assumed that $S^{\alpha\beta}$ is a Legandre first order polynomial, $S^{\alpha 3}$ is the Legandre second order polynomial, and that $S^{33} = 0$, due to the fact that surface limit forces on shell edges are being neglected. The coefficients of Legandre polynomials are determined using limit conditions on shell faces /15/:

$$S^{\alpha\beta} = \frac{1}{2}N^{\alpha\beta} + \frac{3}{2}\zeta M^{\alpha\beta};\ S^{\alpha 3} = S^{3\alpha} = \frac{3}{4}\left(1 - \zeta^2\right)V^\alpha \quad (17)$$

It is now necessary to introduce the following approximation for the volume element dV [16]:

$$dV = \frac{H}{2}\left(1 - \zeta\frac{HB}{2A}\right) dS d\zeta \quad (18)$$

where A is the determinant of the first fundamental form, $A = A_{11}A_{22} - A_{12}{}^2$, and B is repesented by the expression $B = A_{11}B_{22} + A_{22}B_{11} - 2A_{12}B_{12}$.

By placing the stress tensor and volume element in expression [14] and then integrating by coordinate and by using the assumption that γ'_{ab} can be represented by Legandre polynomial, it follows:

$$\begin{aligned}&\gamma'_{\alpha\beta} = \frac{1}{2}\alpha'_{\alpha\beta} + \frac{3}{2}\zeta'_{\alpha\beta}\quad \gamma'_{\alpha 3} = \frac{1}{2}\alpha'_{\alpha 3} + \frac{3}{2}\zeta\beta'_{\alpha 3} + \frac{5}{4}\left(1 - \zeta^2\right)x'_{\alpha 3}\\ &\gamma'_{3\alpha} = \frac{1}{2}\alpha'_{3\alpha} + \frac{3}{2}\zeta\beta'_{3\alpha} + \frac{5}{4}\left(1 - \zeta^2\right)x'_{3\alpha}\quad \gamma'_{33} = \frac{1}{2}\alpha'_{33} + \frac{3}{2}\zeta\beta'_{33}\end{aligned} \quad (19)$$

Finally, it is:

$$\begin{aligned}&\alpha'_{\alpha\beta} = \bar{x}_\alpha^i u_\beta^i + \bar{x}_\beta^i u_\alpha^i\\ &\beta'_{\alpha\beta} = \frac{1}{6}\left(\bar{h}_\alpha^i u_\beta^i + \bar{h}_\beta^i u_\alpha^i + \bar{x}_\alpha^i \kappa_\beta^i + \bar{x}_\beta^i \kappa_\alpha^i\right)\\ &\alpha'_{\alpha 3} = \bar{x}_\alpha^i \kappa^i;\quad \alpha'_{3\alpha} = \bar{h}^i u_\alpha^i \quad x'_{\alpha 3} = x'_{3\alpha} = 0\\ &\beta'_{\alpha 3} = \frac{1}{6}\bar{h}_\alpha^i \kappa^i;\quad \beta'_{3\alpha} = \frac{1}{6}\bar{h}^i k_\alpha^i \quad \alpha'_{33} = \frac{1}{6}\bar{h}^i \kappa^i;\quad \beta'_{33} = 0\end{aligned} \quad (20)$$

Note: Strain measures designated by "′" will be used in the following text but without index "′".

EQUILIBRIUM EQUATIONS

Starting with Cauchy's movement equations in Cartesian coordinates:

$$t_{,j}^{ij} + \rho\left(f^i - \ddot{y}^i\right) = 0 \quad (21)$$

where t^{ij} is the Cauchy stress tensor, f^i and y^i are the volume and inertial forces, respectively, and ρ is the density, the Cartesian coordinates can be transformed into convective curvilenear, neglecting inertial and volume forces obtaining:

$$\left(y_a^i \sigma^{ab}\right)_{|b} = 0 \quad (22)$$

where "|" is the covariant derivative. Further, one can write Eq. [22] in a more convenient form, having in mind the already assumed fact that $\sigma^{33} = 0$.

$$\left(y_\alpha^i \sigma^{\alpha\beta}\right)_{|\beta} + \left(y_\alpha^i \sigma^{\alpha 3}\right)_{|3} + \left(y_\alpha^i \sigma^{3\alpha}\right)_{|3} = 0 \quad (23)$$

In order to derive adequate equilibrium equations for thin shells, Galerkin procedure was applied: Eq. [23] is multiplied by the weight function, for which the rate $\dot{y}^i$ is taken; the equation is integrated on volume dv, which gives the equations used to determine unknown coefficients (node position), appearing in set of base functions, approximatimg the poor solution of the problem.

This gives the following integral:

$$\int\left[\left(y^i_\alpha \sigma^{\alpha\beta}\right)_\beta + \left(y^i_\alpha \sigma^{\alpha 3}\right)_3 + \left(y^i_\alpha \sigma^{3\beta}\right)_\beta\right] \dot{y}^i dv = 0 \quad (24)$$

This integral produces two equilibrium equations in the sense of expression [3]. Expression [18] is used in order to reduce Eq. [24] to a referent configuration. The stress tensor is reduced to referent configuration and since the metric tensor determinant is covariant constant, and so is:

$$\int\left[\dot{x}^i \dot{\phi}^k\right] \int_{-1}^{1} \begin{Bmatrix} 1 \\ \frac{\zeta}{2} e_{ijk} h^j \end{Bmatrix} \left[\left(y^i_\alpha S^{\alpha\beta}\right)\Big|_\beta + \left(y^i_\alpha S^{\alpha 3}\right)\Big|_3 + \left(y^i_3 S^{3\beta}\right)\Big|_\beta\right] \quad (25)$$

$$\left(1 - \zeta \frac{HB}{2A}\right) \frac{H}{2} d\zeta dS = 0$$

Taking into account that Eq. [25] applies to any part of the central surface, one can analyse the integral along the ζ coordinate only, member by member, beginning with:

$$\frac{H}{2} \int_{-1}^{1} \begin{Bmatrix} 1 \\ -\frac{\zeta}{2} e_{ijk} h^j \end{Bmatrix} \left(y^i_\alpha S^{\alpha\beta}\right)_\beta \left(1 - \zeta \frac{HB}{2A}\right) d\zeta \quad (26)$$

The integration on ζ also requires application of involved approximations for y^i_α, Eq. [2a] and $S^{\alpha\beta}$, Eq. [17], whose product is a polynomial of the second order regarding ζ. Taking into account that

$$\left(1 - \zeta \frac{HB}{2A}\right)$$

is independent on θ^α, and that h_j is independent on ζ, one can obtain the adequate expressions for parts of membrane and bending equilibrium equations:

$$\frac{H}{2}\left[x^i_\alpha N^{\alpha\beta} + \frac{1}{2}\left(h^i_\alpha - \frac{HB}{A} x^i_\alpha\right) M^{\alpha\beta}\right]\Big\|_\beta \quad (27)$$

$$-\frac{H}{4} e_{ijk} h^j \left[x^i M^{\alpha\beta} + \frac{1}{6}\left(h^i_\alpha - \frac{HB}{A} x^i_\alpha\right) N^{\alpha\beta}\right]\Big\|_\beta \quad (28)$$

The second member of the starting equation is:

$$\frac{H}{2} \int_{-1}^{1} \begin{Bmatrix} -1 \\ \frac{\zeta}{2} e_{ijk} h^j \end{Bmatrix} \left(y^i_\alpha S^{\alpha 3}\right)_3 \left(1 - \zeta \frac{HB}{2A}\right) d\zeta \quad (29)$$

By applying approximations for y^i_α and $S^{\alpha\beta}$, with direct determination of derivative $(y^i_\alpha S^{\alpha\beta})_3$, integration of Eq. [2] gives:

$$\frac{H}{2}\left(\frac{HB}{2A} x^i_\alpha V^\alpha\right) \text{ – for membrane} \quad (30)$$

$$0 \text{ – for bending part} \quad (31)$$

The third member

$$\frac{H}{2} \int_{-1}^{1} \begin{Bmatrix} 1 \\ -\frac{\zeta}{2} e_{ijk} h^j \end{Bmatrix} \left(y^i_3 S^{3\beta}\right)\Big|_\beta \left(1 - \zeta \frac{HB}{2A}\right) d\zeta \quad (32)$$

$$\text{becomes } \frac{H}{2}\left(\frac{1}{2} h^i V^\beta\right)\Big\|_\beta \quad (33)$$

$$\frac{H}{4} e_{ijk} h^j x^i_\alpha V^\alpha \quad (34)$$

Finally, the membrane and bending equilibrium equations (without inertial, volume and contour surface forces) is:

$$\frac{H}{2}\left\{\begin{aligned}&\left[x^i_\alpha N^{\alpha\beta} + \frac{1}{2}\left(h^i_\alpha - \frac{HB}{A} x^i_\alpha\right) M^{\alpha\beta} + \frac{1}{2} h^i V^\beta\right]\Big\|_\beta \\ &+ \frac{HB}{2A} x^i_\alpha V^\alpha\end{aligned}\right\} = 0 \quad (35)$$

$$-\frac{H}{4} e_{ijk} h^j \left\{\left[x^i_\alpha M^{\alpha\beta} + \frac{1}{6}\left(h^i_\alpha - \frac{HB}{A} x^i_\alpha\right) N^{\alpha\beta}\right]\Big\|_\beta - x^i_\alpha V^\alpha\right\} = 0 \quad (36)$$

CONSTITUTIVE RELATIONS

Linear elasticity of isotropic materials is considered, hence in accordance with the already applied way of deriving thin shell equations, a starting 3D constitutive relation is:

$$S^{ab} = C^{abcd}\gamma_{cd} = \lambda\left[G^{ab}G^{cd} + \mu\left(G^{ac}G^{bd} + G^{ad}G^{bc}\right)\right]\gamma_{cd} \quad (37)$$

where λ and μ are the known Lamé constants. In relation [37] the members in the central shell surface and outside it are separated, already adopted approximate expressions for stress and strain are replaced, and the Galerkin procedure is applied to the expression $S^{ab} - C^{abcd}\gamma_{cd} = 0$. Detailed description is given in /11/ and here the final result is:

$$N^{\alpha\beta} = \frac{E}{1-\nu^2}\left[\nu A^{\alpha\beta} A^{x\psi} + (1-\nu) A^{\alpha x} A^{\beta\psi}\right]\alpha_{x\psi} \quad (38)$$

$$M^{\alpha\beta} = \frac{EH}{3\left(1-\nu^2\right)}\begin{bmatrix}\nu\left(A^{\alpha\beta}B^{x\psi} + A^{x\psi}B^{\alpha\beta}\right) + \\ (1-\nu)\left(A^{\alpha x}B^{\beta\psi} + A^{\beta\psi}B^{\alpha x}\right)\end{bmatrix}\alpha_{x\psi} + \frac{E}{1-\nu^2}\left[\nu A^{\alpha\beta}A^{x\psi} + (1-\nu) A^{\alpha x}A^{\beta\psi}\right]\beta_{x\psi} \quad (39)$$

$$V^\alpha = \frac{2E}{H(1+\nu)} A^{\alpha x}\alpha_{x3} \quad (40)$$

STRAIN ENERGY

The derivation of the expression for specific strain energy is based on its 3D linear elasticity definition:

$$\overline{W} = \frac{1}{2}\int t^{ij}\gamma_{ij} dv \quad (41)$$

which can be transformed into curvilinear coordinates:

$$\overline{W} = \int \begin{aligned}&\frac{1}{2}\sigma^{ab}\gamma_{ab} dv = \\ &\frac{H}{4}\int_S \int_{-1}^{1}\left(S^{\alpha\beta}\gamma_{\alpha\beta} + 2S^{\alpha 3}\gamma^{\alpha 3}\right)\left(1 - \zeta\frac{HB}{2A}\right) d\zeta dS\end{aligned} \quad (42)$$

By applying involved stress and strain approximation:

$$W = \frac{H}{4}\begin{bmatrix}\frac{1}{2}\alpha_{\alpha\beta}N^{\alpha\beta} + \frac{3}{2}\beta_{\alpha\beta}M^{\alpha\beta} - \\ \frac{HB}{4A}\left(\beta_{\alpha\beta}M^{\alpha\beta} + \alpha_{\alpha\beta}N^{\alpha\beta}\right) + \alpha_{\alpha\beta}V^\alpha\end{bmatrix} \quad (43)$$

From Eq. [43] for a thin shell specific strain energy the expressions necessary for the next analysis can be obained:

$$\frac{\partial W}{\partial x^i_\beta}=\frac{H}{4}\left[x^i_\alpha N^{\alpha\beta}+\frac{1}{2}\left(h^i_\alpha-\frac{HB}{A}x^i_\alpha\right)M^{\alpha\beta}+\frac{1}{2}h^i V^\beta\right] \quad (44)$$

$$\frac{\partial W}{\partial h^i_\beta}=\frac{H}{4}\left[x^i_\alpha M^{\alpha\beta}+\frac{1}{6}\left(h^i_\alpha-\frac{HB}{A}x^i_\alpha\right)N^{\alpha\beta}\right] \quad (45)$$

It is to notice that the above results are equal to those parts of equilibrium equations under the covariant differential, so by introducing shortened designanation it follows:

$$R^{i\beta}=\frac{\partial W}{\partial x^i_\beta};\ B^{i\beta}=\frac{\partial W}{\partial h^i_\beta} \quad (46)$$

Therefore, the equilibrium equations can be written as:

$$R^{i\beta}\Big|_\beta+\frac{HB}{2A}x^i_\alpha V^\alpha=0 \quad (47m)$$

$$\frac{1}{2}e_{ijk}h^j\left(B^{i\beta}\Big|_\beta+x^i_\alpha V^\alpha\right)=0 \quad (47b)$$

(m is for membrane, and b is for the bending part).

The derived expressions for strain energy apply to linear elasticity; but, the generality of the approach is not reduced since expressions [46] apply also to nonlinear elasticity.

DERIVING OF THIN SHELL J INTEGRAL CONSERVATION LAW

Using [47m] and [47b] as starting equations which are projected to the tangent plane and integrated on central shell surface, one can obtain:

$$R^{i\beta}\Big|_\beta x^i_\gamma+\frac{HB}{2A}x^i_\alpha V^\alpha x^i_\gamma=0 \quad (48m)$$

$$B^{i\beta}\Big|_\beta h^i_\gamma-x^i_\alpha V^\alpha h^i_\gamma=0\text{, because }\frac{1}{2}e_{ijk}h^j x^k_\gamma=h^i_\gamma \quad (48b)$$

Applying the derivative product rule, Eqs. [44] and [45], the expression for the covariant derivative

$$x^i_\gamma\Big|_\beta=x^i_{\gamma\beta}-x^i_\delta\left\{{\delta \atop \gamma\beta}\right\}\ \text{(same for } h^i_\gamma\big|_\beta\text{)}$$

where $\left\{{\delta \atop \gamma\beta}\right\}$ is the Kristofel symbol, and the following transformations:

$$\frac{\partial W}{\partial x^i_\beta}x^i_\beta=\frac{\partial W}{\partial\zeta^\gamma}=W\Big|_\gamma=\left(W\delta^\beta_\gamma\right)\Big|_\beta\ \text{(same for } h^i\text{), it is:}$$

$$\left(R^{i\beta}x^i_\gamma-W\delta^\beta_\gamma\right)\Big|_\beta+R^{i\beta}x^i_\delta\left\{{\delta \atop \gamma\beta}\right\}+\frac{HB}{2A}a_{\alpha\gamma}V^\alpha=0 \quad (49m)$$

$$\left(B^{i\beta}h^i_\gamma-W\delta^\beta_\gamma\right)\Big|_\beta+B^{i\beta}h^i_\delta\left\{{\delta \atop \gamma\beta}\right\}+\lambda_{\alpha\gamma}V^\alpha=0 \quad (49s)$$

Integrating on central surface and transforming of surface to linear integral according to Gauss will produce:

$$\int_\ell\left(R^{i\beta}x^i_\gamma-W\delta^\beta_\gamma\right)n_\beta d\ell+\int_S\left(W\left\{{\beta \atop \gamma\beta}\right\}+\frac{HB}{2A}a_{\alpha\gamma}V^\alpha\right)d\beta=0 \quad (50m)$$

$$\int_\ell\left(B^{i\beta}h^i_\gamma-W\delta^\beta_\gamma\right)n_\beta d\ell+\int_S\left(W\left\{{\beta \atop \gamma\beta}\right\}+\lambda_{\alpha\gamma}V^\alpha\right)ds=0 \quad (50b)$$

Using standard transformation into a referent configuration for a case of small displacements and strains, delivers:

$$\int_L\left(Wn_1-R^{i\beta}u^i_1 n_\beta\right)dL-\int_S\left(W\left\{{\beta \atop \gamma\beta}\right\}+\frac{HB}{2A}u^i_1 V^i\right)dS=0 \quad (51m)$$

$$\int_L\left(Wn_1-B^{i\beta}\kappa^i_1 n_\beta\right)dL-\int_S\left(W\left\{{\beta \atop \gamma\beta}\right\}-H^i_\alpha\kappa^i_1 V\right)dS=0 \quad (51b)$$

where expressions for $R^{i\beta}$ and $B^{i\beta}$ are linearised

$$R^{i\beta}=X^i_\alpha N^{\alpha\beta}+\frac{1}{2}\left(H^i_\alpha-\frac{HB}{A}X^i_\alpha\right)M^{\alpha\beta}+\frac{1}{2}H^i V^\beta \quad (52m)$$

$$B^{i\beta}=X^i_\alpha M^{\alpha\beta}+\frac{1}{6}\left(H^i_\alpha-\frac{HB}{A}X^i_\alpha\right)N^{\alpha\beta} \quad (52b)$$

The expressions [51m] and [51b] represent the conservation law which applies to thin shells without cracks in which case the strain energy explicit partial derivative (i.e. Laplacian in the general case) for ξ^l coordinate equals zero. If a defect exists then this derivative is not zero, but represents a so called defect force (see /5/), i.e., in this case membrane and bending parts of the thin shell “J” integral can be considered. Quotation marks emphasize that this is not Rice’s expression for J integral, but an analogous expression which applies to thin shells. It should be mentioned that, like in some other situations (thermal strains for example), apart from the linear integral, the additional member also occurs, in the form of a surface integral, which “completes” the linear integral in the sense of its path-independence.

The problem itself is a surface integral analysis [51m] and [51b], and it is of particular interest to determine in which cases the value of these integrals are zero. Without any deeper analysis of this problem, one can conclude that Kristofel symbol will be equal zero for cylindrical shells, and that the transversal shear force will decrease along with shell thickness. Therefore, one can say for linear integrals in expressions [51m] and [51b] that they are parts of cylindrical thin shell J integral, at least in an approximate sense.

Next, the relation between the expression obtained in this way and the stress intensity factor needs to be established. The stress distribution around the crack tip in case of symmetrical loading is given by [19]:

$$N^{\alpha\beta}=\frac{K_m}{\sqrt{2\pi r}}f^{\alpha\beta}(\theta)\quad M^{\alpha\beta}=\frac{K_b}{\sqrt{2\pi r}}f^{\alpha\beta}(\theta)$$

$$f^{11}(\theta)=\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{5\theta}{2}\quad f^{22}(\theta)=\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{2}\cos\frac{5\theta}{2} \quad (53)$$

$$f^{12}(\theta)=f^{21}(\theta)=-\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{5\theta}{2}$$

where K_m is membrane, and K_b is bending stress intensity factor. Using non-dimensional stress intensity factor, which are defined by SIF of adequately loaded plates, it follows:

$$A_m=K_m/K_p\qquad K_p=\frac{pR}{H}\sqrt{\pi a}\qquad\text{longitudinal}$$

$$(54)$$

$$A_b=K_b/K_p\quad K_p=\frac{pR}{2H}\sqrt{2R\pi\,\mathrm{tg}\frac{a}{2R}}\quad\text{circumferential}$$

Finally, the relation between the integral and SIF is:

$$J_m=\frac{H}{E}K_m^2,\ J_b=\frac{H}{3E}K_b^2 \quad (55)$$

RESULTS AND DISCUSSION

The displacement calculation had been conducted using the DSTATA program in a computer centre at the Aeronautical Institute. Linear isoparametric finite elements with four nodes which can simulate transversal shear were used. A special programme was additionally written for calculating J integral which uses the same input data about shell geometry, material and loading, and output displacement data. In order to obtain the results as accurate as possible without the use of special elements, the extrapolating technique was used for results obtained from several finite element regular meshes (a more detailed explanation is found in /20/).

Given as an example is a cylindrical shell under internal pressure with longitudinal and circumferential crack, since the reference data for it are available (e.g. /2/, /18/). The results obtained are given in diagrams of membrane and bending stress intensity factor dependence on shell parameter λ, given by $\lambda = \frac{a}{\sqrt{RH}}\sqrt[4]{12\left(1-\nu^2\right)}$, where $2a$ is the crack length.

The results obtained show a satisfying level of agreement for membrane components of calculated SIF and only partial agreement for bending SIF. In the second case, the disagreement is obvious for greater λ parameter values and is not only related to the bending SIF intensity, but it is also related to its sign. Reference data show the dependence of bending SIF on parameter λ as a function which changes its sign at a certain value of λ, what is not obtained as a result in this work. This difference could be a consequence of introducing the couple of membrane strain and bending stress. The bending SIF is significantly smaller than the membrane, hence their overall SIF is of secondary relevance. More data on this problem will certainly give an example of loading that has bending character, rather than membrane, which is the next step in this work.

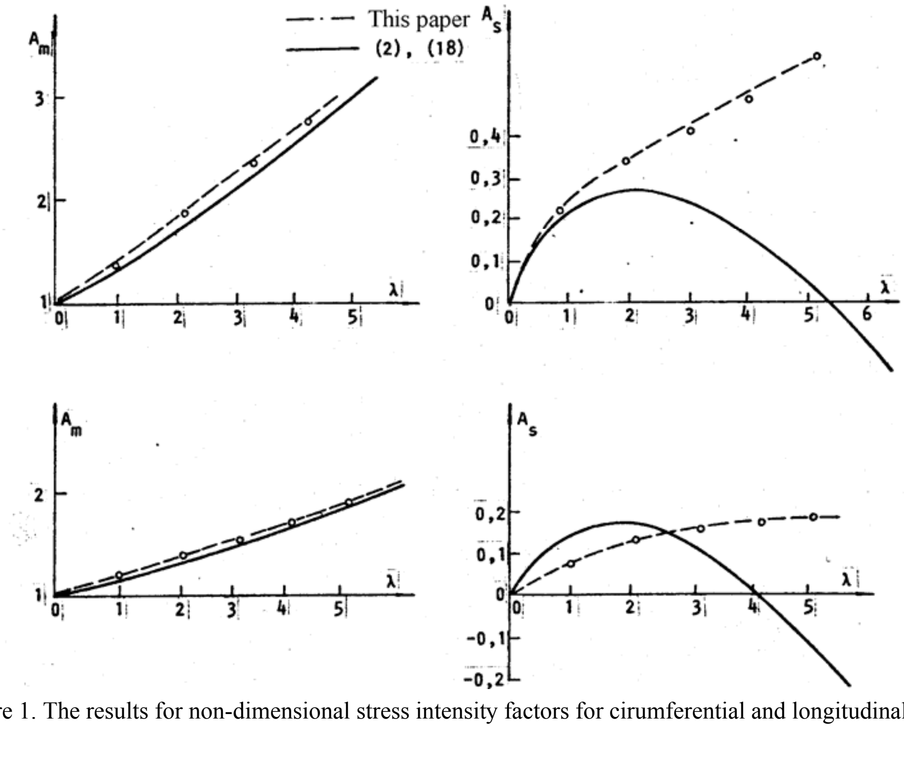

Figure 1. The results for non-dimensional stress intensity factors for cirumferential and longitudinal crack.

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