Third Paper

The original paper (in Serbian) appears in monograph “Modern Aspects of Design and Construction of Pressure Vessels and Penstocks,” edited by S. Sedmak, jointly published by GOŠA Institute (Smederevska Palanka) and the Faculty of Technology and Metallurgy (Belgrade), pp. 35-50, 1982. The monograph contains lectures from the Second International Fracture Mechanics Summer School (IFMASS 2) held in Velika Plana, 1982.

Originalna verzija ovog rada objavljena je na srpskom jeziku u monografiji “Problemi dvoosnog i troosnog stanja napona kod sudova i cevovoda pod pritiskom”, urednik S. Sedmak, zajedničko izdanje Instituta GOŠA (Smederevska Palanka) i Tehnološko-metalurškog fakulteta (Beograd), str. 35-50, 1980. Monografija sadrži predavanja sa Druge međunarodne letnje škole mehanike loma (IFMASS 2) u Velikoj Plani, 1982.

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PROBLEMS OF PLANE AND TRIAXIAL STRESS STATES IN PRESSURE VESSELS AND PIPELINES

Original scientific paper UDC: 62-988:539.421:519.6

INTRODUCTION

This paper has two purposes. First, it gives a short review of basic fracture mechanics terms to readers who meet these problems for the first time, and second, it points out the possibilities of computer analysis in determining the real state of the material around the crack-tip. There is no doubt that the efforts made in order to explain qualitatively the stress state in this zone are sometimes followed by incorrect conclusions. There are more reasons for this – the experimental insight in the stress state, except on the material surface, is still virtually impossible. The possibility of reaching the analytical solution of the problem is limited to plane stress state, linear elasticity, and perfect plasticity. Finally, only a few ways in computer analysis of the triaxial stress state near the crack edge are known up to date.

Taking into account the significant decrease in the price of computer operating hours every year, the triaxial stress state analyses are available to smaller research and development teams. This results in better qualitative and quantitative understanding of stress and strain states in fracture mechanics problems.

Having in mind the purpose of this paper, the attempt will be given to the theoretical basis of fracture mechanics, including the necessary terms of elasticity and plasticity theories, in a short, yet comprehensive way. Also, some results of computer analysis using finite element method will be presented.

FUNDAMENTAL TERMS IN SOLID BODY MECHANICS

Kinematics basis

As usual, Cartesian coordinates of a point in a solid body, also known as radius vector coordinates, will be written as:

$$z^i \ (i = 1,2,3) \tag{1}$$

The displacement vector represents the difference between the positions of the current and the starting configuration, and its coordinates are written as:

$$u_i = u_i(z^i,t) \tag{2}$$

Derivatives of displacement vector along the coordinates are called displacement gradients,

$$u_{i,j} = \frac{\partial u_i}{\partial z_j} \tag{3}$$

The theory of elasticity shows that if displacement gradients are small compared to the unit the strain tensor components can be presented as:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{4}$$

Stress state

The effect of a force on a unit of randomly oriented area inside or on the surface of the observed body can be represented by the stress tensor σ_{ij}. One can consider stress tensor components as forces which act along the z^i direction to the unit area with a normal in the z_j direction. In continuum mechanics, it can be shown that $\sigma_{ij} = \sigma_{ji}$, therefore the stress tensor has 6 different components.

The terms: principal stress σ_a, and principal direction n_i^a are of great importance for the study of plane and triaxial problems in fracture mechanics. These quantities respectively represent eigenvalues and eigenvectors of problems:

$$\left[\sigma^{ij} - \sigma\delta^{ij}\right] n_i = 0 \tag{5}$$

where $\delta^{ij} = 1$ for $i = j$, and $\delta^{ij} = 0$ for $i \neq j$, represent the Kronecker delta symbol. The principal directions of n_i are perpendicular to each other. In planes whose normal bisects principal angles between the principal directions, maximum shear stress occurs, given by:

$$\tau_1 = \frac{\sigma_2 - \sigma_3}{2},\ \tau_2 = \frac{\sigma_3 - \sigma_1}{2},\ \tau_3 = \frac{\sigma_1 - \sigma_2}{2} \tag{6}$$

assuming that:

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \tag{7}$$

then the largest shear stress will occur in two planes at an angle of 45° to the directions of stresses σ_1 and σ_3 (Fig. 1).

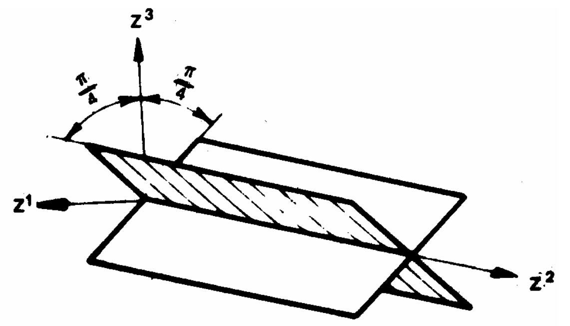

Figure 1. Planes of maximum shear stress.

By expanding the determinant with coefficients from [5], an equation of the third order is obtained:

$$\sigma^3 - I_{1_\sigma}\sigma^2 + I_{2_\sigma}\sigma - I_{3\sigma} = 0 \tag{8}$$

whose coefficients are the invariants of stress tensor:

$$I_{1_\sigma} = \sigma_1 + \sigma_2 + \sigma_3,$$
$$I_{2_\sigma} = \sigma_1\sigma_2 + \sigma_3\sigma_1 + \sigma_2\sigma_3, \tag{9}$$
$$I_{3_\sigma} = \sigma_1\sigma_2\sigma_3$$

The term stress deviator is significant in the theory of plasticity:

$$\tilde{\sigma}^{ij} = \sigma^{ij} - \frac{1}{3} I_{1_\sigma}\delta^{ij} \tag{10}$$

Invariants of stress deviator are as follows:

$$I_{1_{\tilde{\sigma}}} = 0,$$
$$I_{2_{\tilde{\sigma}}} = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \tag{11}$$
$$I_{3_{\tilde{\sigma}}} = \tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_3$$

Also, the term of effective stress (Mises stress) is common in the theory of plasticity:

$$\Sigma = +\sqrt{3I_{2_{\tilde{\sigma}}}} \tag{12}$$

Relations between stresses and strains

For an isotropic linear elastic material, the general relation between stress and strain is as follows:

$$\sigma^{ij} = \frac{2\mu}{1-2\nu}\left[\nu\delta^{ij}\delta^{kl} + (1-2\nu)\delta^{ik}\delta^{jl}\right] e_{kl} \tag{13}$$

where μ is the shear modulus and ν is Poisson's ratio.

If effects of plastic strain are significant for a given problem, one of the plasticity theories will be used. These theories are mostly too complex from engineering points of view. However, for a practically relevant case of simple loading (the case in which the external forces increase proportionally to a single parameter), it can be shown that it is possible to present a plastic body by a nonlinear elastic model. This approach is also called the deformation theory of plasticity. Therefore, in case of simple material loading, the relation [13] can be used even in a plastic range. The computer programmes based on [13] can also be used. The only difference is that instead of constants μ and ν in [13], variable parameters are introduced:

$$\mu_S = \frac{E_S}{2(1+\nu_S)} \tag{14}$$

$$\nu_S = \frac{1}{2}\left[1 - \frac{E_S}{E}(1-2\nu)\right] \tag{15}$$

where E_S is the secant elasticity modulus, while E is the elasticity modulus of an unloaded material. Both quantities are obtained from experimentally determined σ–ε uniaxial stress–strain diagram.

While examining some materials (such as mild steels) under uniaxial stress, one can notice that up to a certain value of stress, these materials show linear elastic behaviour, and then, for a constant stress value, the material deforms with no limit (yield) up to fracture. This stress is known as yield stress. For the purpose of standardizing, the stress $\sigma_T = \sigma_{0.2}$, corresponding to relative strain $\varepsilon = 0.2\%$ under uniaxial stress is taken as the yield stress. The above definition of yield stress allows to expand this term to materials with an extremely smooth σ–ε curve (e.g. aluminium alloys).

This behaviour opens a question how to determine yield stress under a complex stress state. As a result of generalizing the existing experimental data, more hypotheses have been suggested /2/.

One of the best-known is the Tresca criterion which suggests that yielding occurs when at least one of the maximum shear stresses reach one half of the values of yield stress, under uniaxial stress. Hence:

$$2|\tau_1| = |\sigma_2 - \sigma_3| \leq \sigma_T,$$
$$2|\tau_2| = |\sigma_3 - \sigma_1| \leq \sigma_T, \tag{16}$$
$$2|\tau_3| = |\sigma_1 - \sigma_2| \leq \sigma_T$$

Especially in a three-dimensional case, these conditions are unpractical and are therefore replaced with the following yield criterion of Mises:

$$\Sigma = \sigma_T \tag{17}$$

Plane strain

The plane strain state corresponds to a strain of a long cylinder exposed to uniform and equal forces lying in all planes perpendicular to the cylinder's directrissa (Fig. 2).

It is obvious that under plane strain state the displacement is a function of plane coordinates. There is no displacement or strain along the z_3 (cylinder axis), therefore:

$$u_\alpha = u_\alpha(x_\beta) \quad \alpha,\beta = 1,2 \quad u_3 = 0 \quad \varepsilon_{33} = 0 \qquad (18)$$

Also, the shear stresses except those in the considered planes are equal

$$\sigma^{23} = \sigma^{31} = 0 \qquad (19)$$

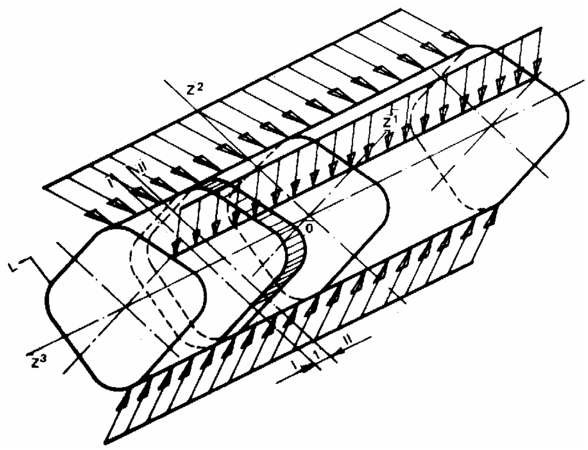

Figure 2. Plane strain state.

The stress σ^{33} in the direction of the cylinder axis is one of the principle stresses and its calculation is based on the known plane stresses:

$$\sigma^{33} = \nu\left(\sigma^{11} + \sigma^{22}\right) \qquad (20)$$

The maximum shear stress can be determined from:

$$\tau_{\max} = \frac{1}{2}\sqrt{\left(\sigma^{11} - \sigma^{22}\right)^2 + 4\left(\sigma^{12}\right)^2} \qquad (21)$$

The two-dimensional model is essential for considering the stress state in the vicinity of the crack edge, because in a plate of finite thickness the edge represents a long tunnel. Near the tunnel, the stress state can be qualitatively represented at least by using plane strain equations.

Plane stress

Plane stress (plane stress state) is such a stress state of a plate in which it is loaded by forces in its plane (Fig. 3).

Equations [19] are also applicable here. However, unlike the case of plane strain, where σ^{33} is determined from [20], here one of the principal stresses is:

$$\sigma_3 = \sigma^{33} = 0 \qquad (22)$$

Further, the Mises stress [12] is reduced to:

$$\Sigma = +\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \qquad (23)$$

Taking into account $\sigma_3 = 0$, the maximum shear stress can be determined in the case that σ_1 and σ_2 are of the same sign, from the following equation:

$$\tau_{\max} = \frac{1}{2}\left|\sigma_1 - \sigma_2\right| \qquad (24)$$

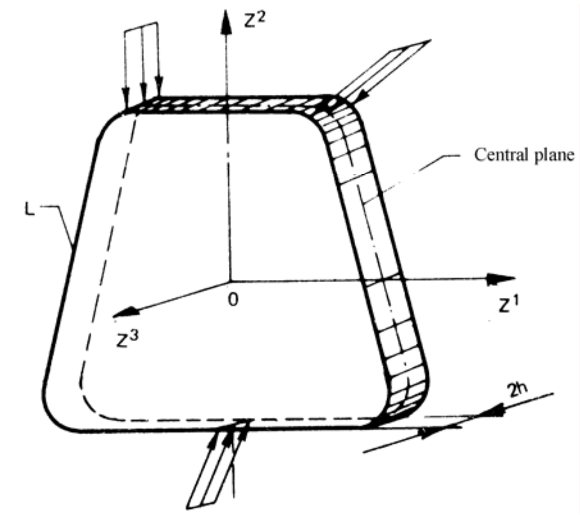

Figure 3. Plane stress state.

In case that σ_1 and σ_2 are of different signs, assuming $|\sigma_1| > |\sigma_2|$, the maximum shear stress will be:

$$\tau_{\max} = \frac{1}{2}\sigma_1 \qquad (25)$$

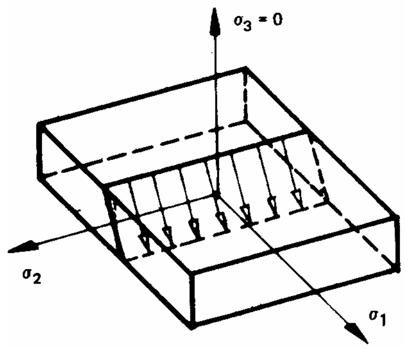

Figure 4. Maximum shear stress.

This stress acts in a plane which forms an angle of 45° to the direction of stresses σ_1 and σ_3.

FUNDAMENTAL TERMS IN FRACTURE MECHANICS

Theoretical material strength

By experimentally analysing interatomic forces in a crystal lattice /1/, one can conclude that the stress required to separate two adjacent atomic layers equals:

$$\sigma_o \approx 0.1E \qquad (26)$$

where E is the elasticity modulus. However, because of the existence of cracks in the material, the real material strength for brittle material (such as glass) is given in case of plane strain by the Griffith formula:

$$\sigma_C = \sqrt{\frac{2E_\gamma}{\left(1-\nu^2\right)\pi a}} \qquad (27)$$

where

$$\gamma \approx 0.01Er_o \qquad (28)$$

is the density of surface energy (work per unit of free area), the amount of work necessary to overcome interactive forces between adjacent layers of atoms, and r_o being the distance between them. The existing crack length in the material is $2a$.

By examining metal it is determined that it does not behave according to Griffith's formula due to the plastic strain near the crack tip. Now the expression [27] is being expanded so it would take this effect into account:

$$\sigma_C = \sqrt{\frac{2E\left(\gamma + \gamma_p\right)}{\left(1 - \nu^2\right)\pi a}} \tag{29}$$

where γ_p is the work of plastic deformation needed to form a new unit crack area, i.e. γ_p for steel is 103γ.

Stresses and displacements near the crack tip in a linear elastic material

In a plane problem of the elasticity theory, the modes of fractures can be classified as presented in Fig. 5.

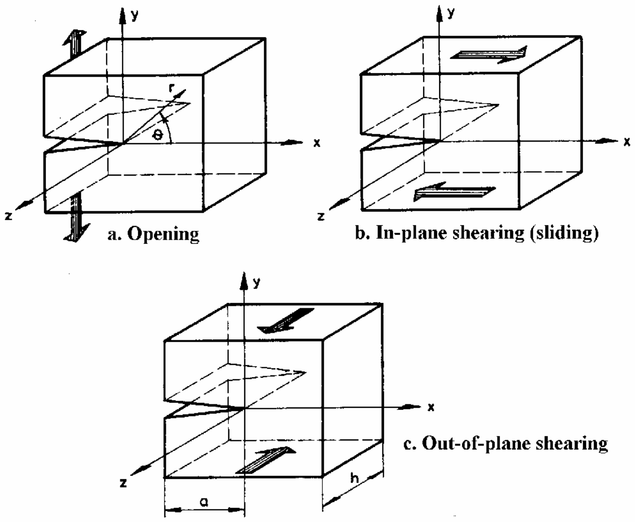

Figure 5. Modes of fracture development.

For these three fracture modes the stress state near the crack tip can be described by the following expression:

$$\sigma^{ij} = \frac{K_L}{\sqrt{2\pi r}} f_{ij}^{(L)}(\theta) \tag{30}$$

The displacements are:

$$u_i = \frac{K_L}{2\mu}\sqrt{\frac{r}{2\pi}} f_i^{(L)}(\theta) \tag{31}$$

In the equations above, the coefficients K_L are stress intensity factors. Functions $f_{ij}^{(L)}$, $f_i^{(L)}(\theta)$ depend only on the angle θ and the fracture mode. For a homogenous stress state at a sufficient distance from the crack tip, the stress intensity factors are:

$$K_L = K_{\mathrm{I}} = \sigma\sqrt{\pi a} \tag{32}$$

for cleavage, and

$$K_L = K_{\mathrm{II}} = K_{\mathrm{III}} = \sigma\sqrt{\pi a} \tag{33}$$

for slide and shear.

Plastic behaviour near the crack tip

From Eq. [5] it follows that the stress near the crack tip is reversely proportional to the square root of the radius, hence it grows infinitely approaching the crack tip. This theoretical result does not comply with real material behaviour which is, in case of high stress values, best described by the plasticity theory.

From the physics point of view, plastic yielding appears due to the sliding of adjacent layers over each other. This occurs when the shear stress reaches material yield stress. Therefore, it is clear that yielding will occur in the direction of maximum shear stress. In case of plane stress this assumes yielding at an angle of 45° to the plane's normal, and the direction of the greater of the two principal stresses, (Fig. 4).

In case of plane strain perpendicular to the x_3 axis, for $0 < \sigma_2 < \sigma_3$, $\sigma_2 < \sigma_1$, the maximum shear stress will occur at yield stress $\sigma_1 - \sigma_2 = \sigma_T$, therefore:

$$\sigma_1 = \sigma_T + \sigma_2 > \sigma_T \tag{34}$$

Because of this, the stress σ_1 under plane strain can be significantly above yield stress without yielding. As a consequence, in case of plane strain there is no fracture by slide in planes at 45° to the direction of the force. Instead, cleavage occurs, separating adjacent layers of atoms in planes perpendicular to directions of external loading without any significant plastic strains, /3/.

The stress intensity factor value at which fracture occurs:

$$K_{\mathrm{I}c} = \sigma_K\sqrt{\pi a} \tag{35}$$

is called fracture toughness. If the stress field is homogenous on a greater distance from the crack tip, σ_K will be the value of stress perpendicular to the crack under which fracture occurs.

The size of the plastic zone is determined assuming that the stress within this zone is σ_T, while the stress intensity factor equals K_{I}. For $\theta = 0$, Eq. [35] gives:

$$r = \frac{1}{2\pi}\left(\frac{K_{\mathrm{I}}}{\sigma_T}\right)^2 \tag{35}$$

The result of [35] complies with more specific considerations of plane stress. However, in case of plane strain, the size of the plastic zone is very small and can be approximately determined from:

$$r \approx \frac{1}{6\pi}\left(\frac{K_{\mathrm{I}}}{\sigma_T}\right)^2 \tag{36}$$

Stress state in a specimen of finite thickness

Strictly speaking, plane stress occurs only in an infinitely thin plate, and plane strain occurs only in an infinitely thick plate. On the other hand, in a plate of finite thickness, it can be expected that the plastic zone is within the limits given by [35] and [36].

Based on the shape of the plastic zone, calculated separately for plane stress and plane strain, the plastic zone shape in Fig. 6 is assumed using interpolation /3,4/.

However, recent numerical analyses make this popular interpretation of plastic zone shape questionable. For example, Fig. 7 shows a plate with a central symmetrical crack, with a mesh of finite elements, while the size and shape of the plastic zone are shown in Fig. 8, /5/.

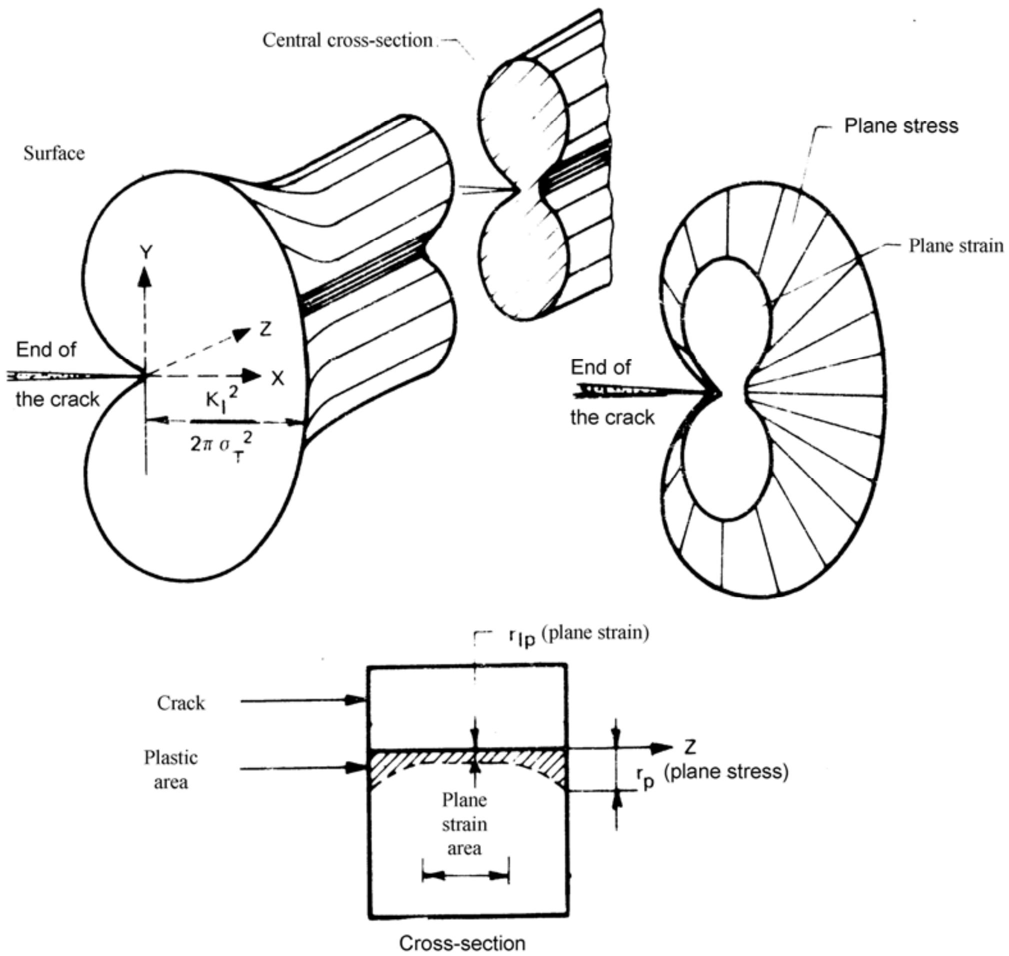

Figure 6. Plastic area around the crack tip.

The difference between plastic zone scales on the contour and in the plate's symmetry plane is negligible.

Similar results were obtained using three-dimensional analysis of a compact specimen /6/. The shape of the plastic zone reminds of the plane strain case, while its size is approximately equal to the case of plain stress (Fig. 9). Diagrams in Fig. 10 allow a better insight of stress state. One can conclude from curve 1 in Fig. 10 that the stress σ_Z – in the direction perpendicular to the side surface of specimen, behaves as expected, meaning that it changes from a relatively large value to zero in the contour, as limiting conditions suggest. However, three-dimensional analysis gives a result which is impossible to extrapolate from two-dimensional analysis – curve 2 in Fig. 10 represents the distribution of shear stress σ_{xz} along specimen thickness. This stress, necessary for balancing the variable σ_z stress along specimen thickness, does not appear in two-dimensional analysis, although it has a significant effect on forming of the plastic zone. Hence, the effective stress Σ [12] practically does not change at all along specimen thickness (curve 3 in Fig. 10).

However, as thickness increases, the shear stresses σ_{xz} (curve 2) necessary to balance σ_z, are smaller and smaller, considering that this balancing takes place at a greater distance between the symmetry plane and the contour, hence the stress state tends to the state of plane strain. Such reasoning is confirmed by numerical results, /6/.

Given results justify the analysis of plane strain state testing using specimens of finite thickness. Of course, this brings up a question of specimen thickness required, so that the stress state has no significant differences compared to the plane strain state.

Generalizing a large number of experimental data has resulted in a conclusion that in case of ratio $r/H < 0.02$, where r is the radius of the plastic zone, and H is specimen thickness, fracture toughness testing results correspond to the state of plane strain, /7/. By placing the value r in [36], the requirement for specimen thickness can be obtained

$$B > 2.5\left(\frac{K_{\mathrm{I}c}}{\sigma_T}\right)^2 \qquad (37)$$

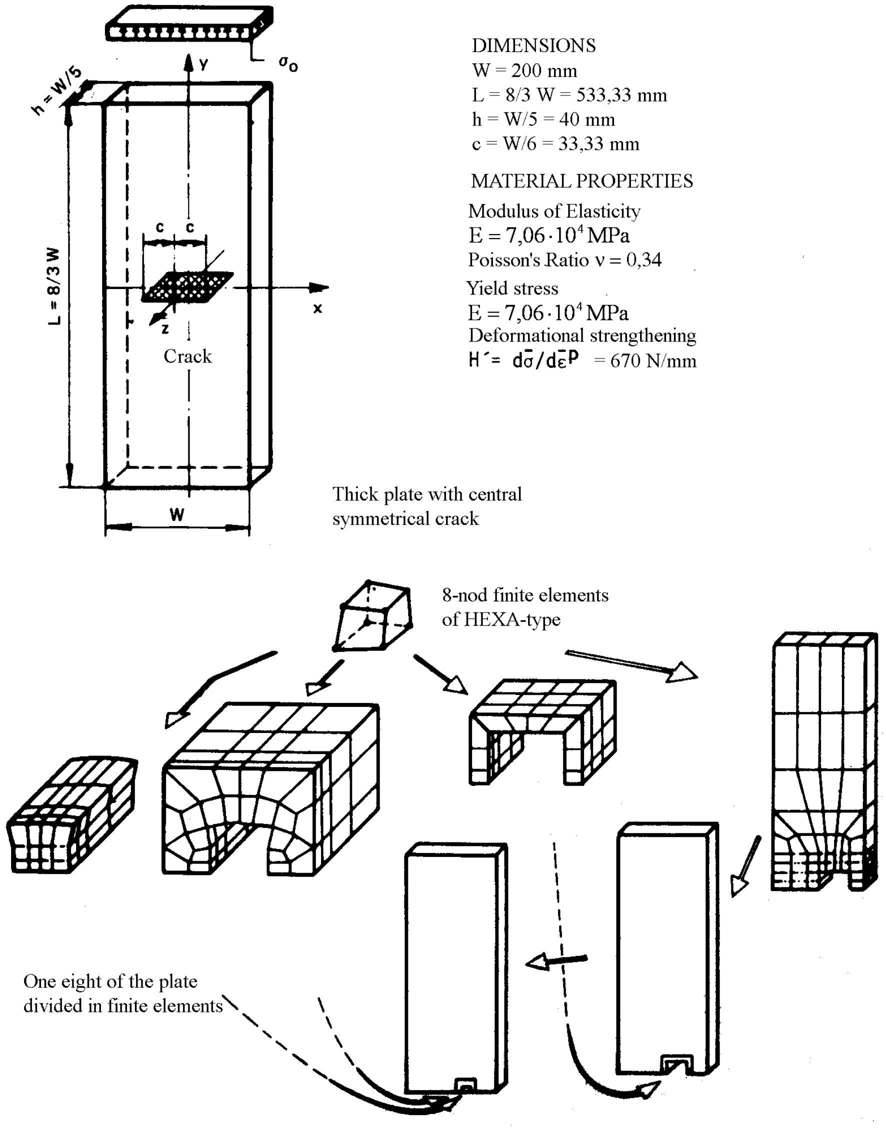

Figure 7. A plate with central crack, divided into finite elements.

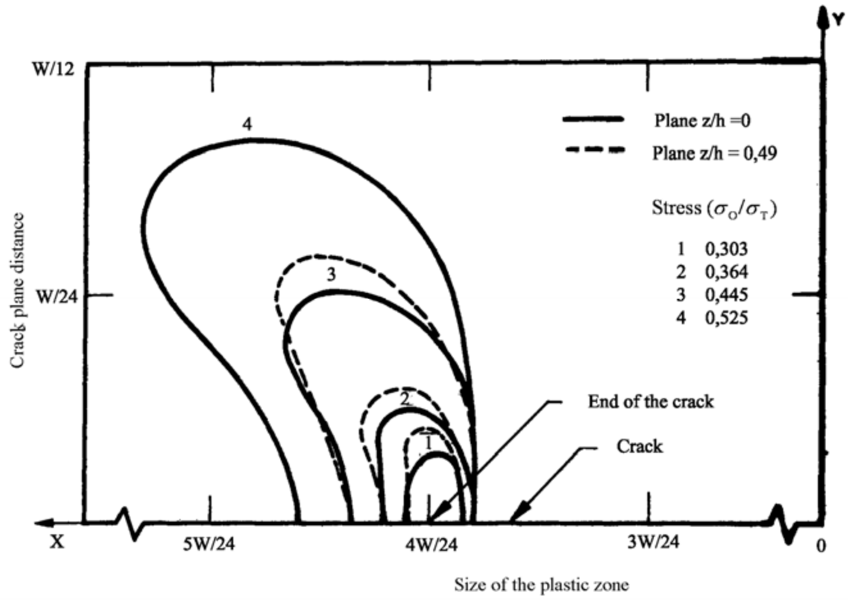

Figure 8. Size and shape of the plastic zone of the panel from Fig. 7.

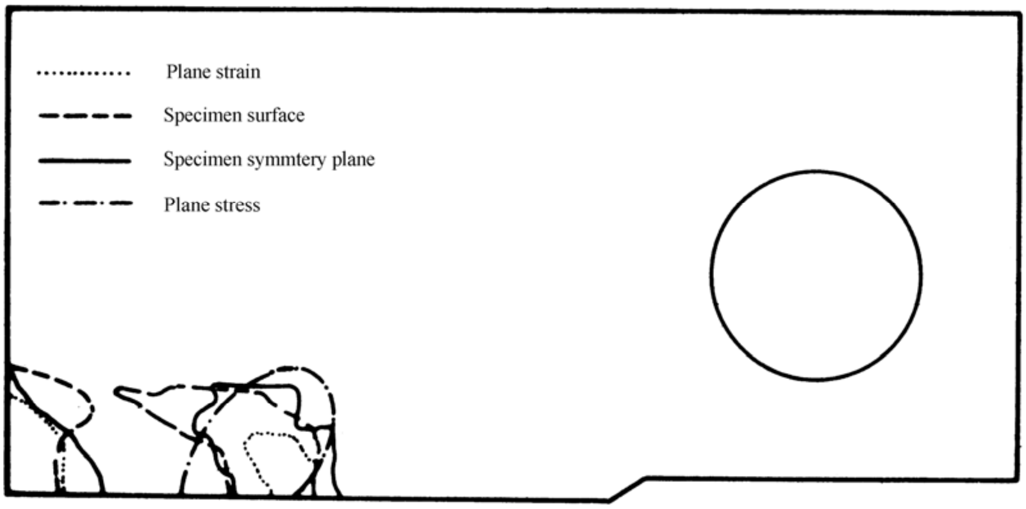

Figure 9. Comparison of plastic zones for plane strain and the centre of specimen, for plane stress and specimen surface.

GENERALIZATION OF THE RESULT TO PROBLEMS OF PRESSURE VESSELS AND PIPELINES

In case of pressure vessels and pipelines, previously obtained results can be generalized having in mind the geometry characteristics and loadings for this type of structure (primarily curvature and pressure). The diagram in Fig. 10 shows the obvious effect of curvature radius to vessel thickness ratio on the stress intensity factor /8/. This problem is a subject of numerous recent researches, therefore some relevant results are given in the Proceedings book.

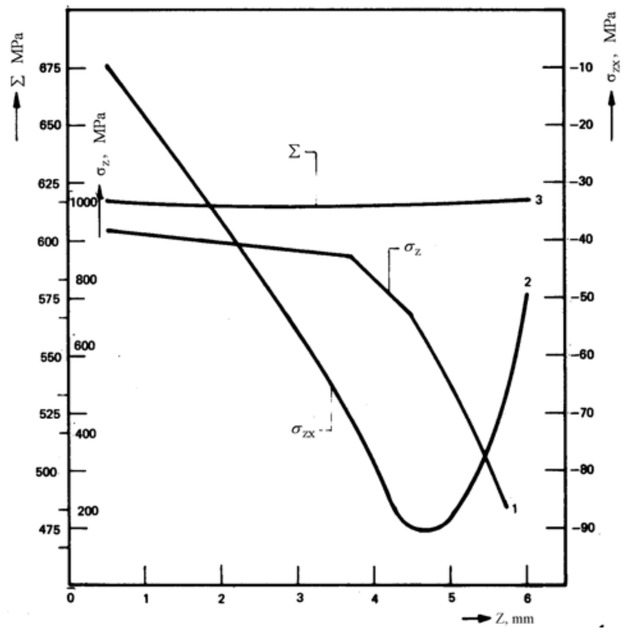

Figure 10. The change of characteristic stresses from central plane to surface.

DISCUSSION AND CONCLUSIONS

The problem of fracture mechanics in various applications including pressure vessels and pipelines is basically three-dimensional (tri-axial). Depending on object thickness and shape of the crack, certain simplifications are possible and reduction to two-dimensional problems of plane stress or three-dimensional problems of plane strain. However, the results of such simplifications should not be taken for granted because existing three-dimensional analysis shows not only quantitative, but also qualitative deviations from simplified schemes.

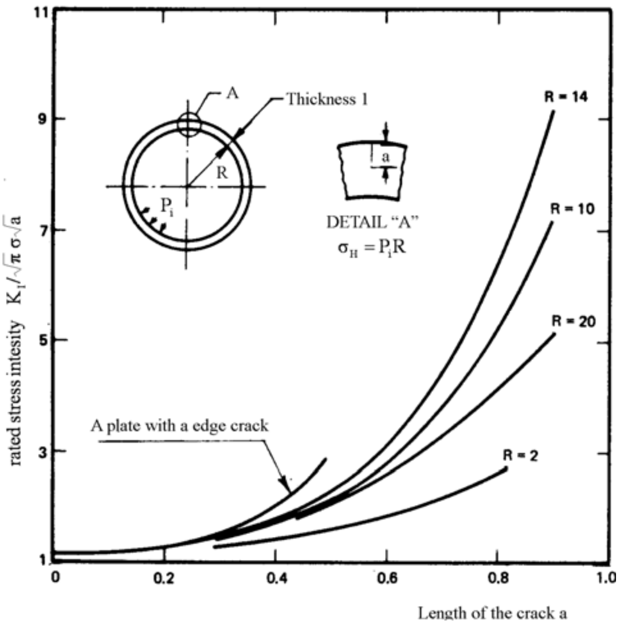

Figure 11. Change of stress intensity factors with crack entail and vessel radius.

Fortunately, thanks to the powerful development of numerical analysis and computer technology during the last decade, three-dimensional analysis of fracture mechanics problems became a routine not only for well-equipped research teams, but also for medium sized design organisations with expanding tendencies, and for smaller organisations. Therefore, the key problem of design today is how to master computer techniques. Some very competent thoughts on this have been presented in the book of Proceedings.

Of course, the physical nature of the problem also considered here can never be neglected.

Last but not least, the result of an experiment remains the final evaluation of theory and calculation.

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