Second Paper

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NUMERICAL METHODS IN FRACTURE MECHANICS

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From the text presented in previous chapters, we can conclude that the fundamental problem in fracture mechanics is the assessment of stress distribution near the sharp crack tip. Stress distribution is determined as well as in solid body mechanics in general, by analytical and numerical methods. Some experimental methods are described in Chapter 4 with emphasis on standard specimen testing. Analytical methods described in Chapters 2 and 3, apart from the qualitative insight of fracture phenomenology, enable obtaining an accurate solution for a certain number of simple problems. Numerical methods represent the only acceptable alternative for most fracture mechanics applications due to the fact that it is related to the solid body of highly complex forms. Despite the abundance of numerical methods, from which many are suitable for solving certain fracture mechanics problems, in this paper we will focus on the finite element method. The reason for this lies in the generality and flexibility of this method which, combined with application of up-to-date computers, has lead to an industrial revolution in the field of numerical calculation and design in general.

APPLICATION OF FINITE ELEMENT METHOD IN DETERMINING STRESS AND DISPLACEMENT AROUND THE CRACK TIP

First of all, we should emphasize that in fracture mechanics, general finite element method programmes are being used, and that there are no differences between the finite element method and the other solid-body mechanics problems when defining and solving equations as well as in selecting material models. When applying this method in elastic /1,2/ and elastic-perfectly plastic /3/ problems, special finite elements with correct stress and strain singularity are used. However, in the general non-linear case, such approach is pointless and not very useful for special problems, hence we will not consider it any further. Instead, we will consider the effective possibilities of determining certain fracture mechanics parameters by using standard software, which is more or less available to every finite element method user.

As we know, the application of this method involves, above all, the forming of an adequate mesh in which the observed object is divided into a number of topologically uniform elements. In case of standard finite element method programme application on fracture mechanics problems, a significant refinement of the mesh around the crack edge is necessary (see First Paper, Fig. 3).

Application of this method on a considered problem results in displacement of mesh nodes and hence in the determination of stresses. Once these quantities are known, we can determine the fracture mechanics parameters by using Eqs. A1 to A10, given in the Appendix (most of the Eqs. given here are cited from original paper Proceedings, please use this reference if necessary).

Determining stress intensity factors based on given stress values

Considering Eqs. A1, A3 and A7, we can conclude that for given coordinates r and θ, and stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{xz} and τ_{yz}, the parameters K_I, K_{II} and K_{III} are solvable. For practical reasons we will assume that $\theta = 0$, which results from these equations, in respect:

$$K_{\mathrm{I}} = \sigma_y\sqrt{2\pi r}\ ,\ K_{\mathrm{II}} = \tau_{xy}\sqrt{2\pi r}\ ,\ K_{\mathrm{III}} = \tau_{yz}\sqrt{2\pi r} \quad (1)$$

When effectively determining these values, we usually calculate several of them along the direction θ = const, and then evaluate the most convenient among them by extrapolating for $r = 0$ (First Paper, Fig. 6). Such a procedure, although simple, requires a certain amount of time and effort in graphical elaboration of results.

Determining stress intensity factors based on given displacement values

This procedure is analogous to the above and expressions used are A2, A4 and A6, respectively. From practical reasons, we will assume that $\theta = \pi$, which means that we will observe displacements on the crack surface, hence:

$$K_{\mathrm{I}} = \frac{\mu u_1}{\kappa+1}\sqrt{\frac{2\pi}{r}}\ , K_{\mathrm{II}} = \frac{\mu u_2}{\kappa+1}\sqrt{\frac{2\pi}{r}}\ , K_{\mathrm{III}} = -\mu u_3\sqrt{\frac{\pi}{2r}} \quad (2)$$

Generally, this approach is a bit more accurate then the previous due to higher displacement accuracy compared to the stresses when using FEM.

Determining stress intensity factors based on energy release rate under crack growth

When this approach is being used, stress intensity factors may be explicitly determined only for pure strain forms (see First Paper, Fig. 1); e.g. if $K_{\mathrm{II}} = K_{\mathrm{III}} = 0$ it follows in accordance with /4/ that:

$$K_{\mathrm{I}} = \sqrt{\frac{8\mu}{\kappa+1}G} \quad (3)$$

according to A9 and A10. Concerning the strain energy release rate G itself, we will determine it by using two consecutive finite element method analyses, for two close crack lengths which differ by Δa. Based on /5/ we can write approximately:

$$G = -\frac{dU}{h da} = -\frac{\Delta U}{h\Delta a}$$

where U/h represents strain energy per unit thickness h, for plane stress state, and $\Delta U = U_2 - U_1$ is the difference of strain energies for two crack of close lengths under the same external loading. In case of plane strain, $h = 1$. Further, in case of solving static problems in linear elasticity theory by using the finite element method, the strain energy equals one half of external force work, with a reverse sign:

$$U = -\frac{1}{2}R^T u$$

where the n-dimensional vector R designates external forces affecting the structure, and vector u represents their adequate displacements. The disadvantage is the need to conduct two analyses or some complicated interventions in the program /6/.

Determining stress intensity factors based on Rice's J integral

Similarly to Eq. [3] we can write:

$$K_{\mathrm{I}} = \sqrt{\frac{8\mu}{\kappa+1}J} \quad (4)$$

The problem is reduced to calculating the J integral, which is given in Eq. A8 in its vector form. Rice's J integral is the component of A8, which can also be shown with the following expression /5/:

$$J = \int_\Gamma \left(W\,dx^2 - \sigma^{ij} n_j \frac{\partial u_i}{\partial x^1} ds \right) \quad (5)$$

The use of Eq. [5] requires the crack to lie in the x^3x^1 plane. For a linear elastic material with a symmetrical stress tensor, the strain energy per unit volume is:

$$W = \frac{1}{2}\sigma^{ij}u_{i,j} \quad (6)$$

where $u_{i,j}$ denotes the displacement gradients:

$$u_{i,j} = \frac{\partial u_i}{\partial x^j} \quad (7)$$

Taking into account that the contour surface is parallel to the x^3 axis, the contour's normal vector coordinates are:

$$n_1 = \frac{dx^2}{ds}\ ,\ n_2 = -\frac{dx^1}{ds} \quad (8)$$

Hence the index j in Eqs. [5], [6] and [7] takes the values $j = 1,2$, while $i = 1,2,3$. The previous statement does not conflict with the assumption of stress and strain homogeneity along the x^3 axis. It also allows us to treat the shear strain (see First Paper, Fig. 1) by using the J-integral.

In most general cases of a linear elastic material which applies to plane strain, the constitutive equations are given in the following form:

$$\sigma^{ij} = \frac{2\mu}{1-2\nu}\left[\nu\delta^{ij}\delta^{kl} + (1-2\nu)\delta^{ik}\delta^{jl}\right]u_{(k,l)} \quad (9)$$

In case of plane strain, this expression is reduced to:

$$\sigma^{ij} = \frac{2\mu}{1-\nu}\left[\nu\delta^{ij}\delta^{il} + (1-\nu)\delta^{ik}\delta^{jl}\right]u_{(k,l)} \quad (10)$$

where $u_{(k,l)} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right)$. In both expressions $j = 1,2$ and k, l, $i = 1,2,3$ with the condition that $u_{k,3} = 0$. By placing [6], [8] and [9] or [10] into [5] and using some algebra, we obtain the expression for J integral under plane and anti-plane strain or stress as:

$$J = \frac{\mu}{2}\int_\Gamma \left\{ \begin{array}{l} \left[\kappa_1\left(u_{2,2}^2 - u_{1,1}^2\right) + u_{1,2}^2 - u_{2,1}^2 + u_{3,2}^2 - u_{3,1}^2\right]dx^2 + \\ +2\left(u_{1,1}u_{1,2} + u_{3,1}u_{3,2} + \kappa_2 u_{1,1}u_{2,1} + \kappa_1 u_{2,1}u_{2,2}\right)dx^1 \end{array} \right\} \quad (11)$$

where, under anti-plane strain:

$$\kappa_1 = \frac{2(1-\nu)}{1-2\nu}\ ,\ \kappa_2 = \frac{1}{1-2\nu} \quad (12)$$

as for plane and anti-plane stress, we have:

$$\kappa_1 = \frac{2}{1-\nu}\ ,\ \kappa_2 = \frac{1+\nu}{1-\nu} \quad (13)$$

The Eq. [11] is suitable for numerical calculation, since variables in it are only displacement gradients. However, these may be determined based on given displacement fields in an element or, if these are not known, by using the finite difference method /4/. Having in mind the great accuracy of displacements in nodes determined with FEM as well as the generally irregular network, the Hudec /7/ method can be useful for determining displacement gradients. An elementary case will be taken for presentation. Let nodes J and K lie on a contour Γ, and L is the node closest to J and not collinear to the above two. The displacement gradients, e.g. u_3 in point J, are approximately:

$$u_{3,1} = \frac{u_{3KJ} x_{LJ}^2 - u_{3LJ} x_{KL}^2}{x_{KJ}^1 x_{LJ}^2 - x_{LJ}^1 x_{KJ}^2}\,,\ u_{3,2} = \frac{u_{3LJ} x_{KJ}^1 - u_{3KJ} x_{LJ}^1}{x_{KJ}^1 x_{LJ}^2 - x_{LJ}^1 x_{KJ}^2} \quad (14)$$

where $u_{3KJ} = u_{3K} - u_{3J}$ represents the difference between displacements of points K and J. In an analogous way we form coordinate differences x^2_{LJ}, etc. Also in the integral [11], we will replace dx^1 with x^1_{KJ} and dx^2 with x^2_{KJ}, and replace the sub-integral functions:

$$\begin{aligned} F_1 &= \kappa_1 \left(u_{2,2} + u_{1,1}\right)\left(u_{2,2} - u_{1,1}\right) + \\ &\quad + \left(u_{1,2} + u_{2,1}\right)\left(u_{1,2} - u_{2,1}\right) + \\ &\quad + \left(u_{3,2} + u_{3,1}\right)\left(u_{3,2} - u_{3,1}\right) \\ F_2 &= 2\left[u_{1,1}u_{1,2} + u_{3,1}u_{3,2} + u_{2,1}\left(\kappa_2 u_{1,1} + \kappa_1 u_{2,2}\right)\right] \end{aligned} \quad (15)$$

with their mean values F_{1KJ} and F_{2KJ} on the segments of Γ. In this case the integral [11] is reduced to a sum along segments of the considered contour:

$$J = \frac{\mu}{2} \sum_{J,K=1,2}^{J,K=N-1,N} \left(F_{1KJ} x_{KJ}^2 + F_{2KJ} x_{KJ}^1\right) \quad (16)$$

The example of fracture mechanics parameter determination for a plate with a central symmetrical crack

We assume that the plate is in a state of plane stress. The adopted mesh is shown in Fig. 1 of the First Paper. The same figure also shows very good agreement of stress results with the known analytical solution by Vestergard.

Further, in Fig. 3 of the First Paper the iso-lines of Mises stresses are shown:

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} \quad (17)$$

where σ_1 and σ_2 are main stresses near the crack tip.

Figure 4 in the First Paper shows a deformed configuration near the crack tip, magnified for its better presentation.

Figure 5 in the First Paper shows the graphical determination of stress intensity factors based on displacements and stresses that were calculated using the standard package module SMS, developed in the Aeronautical Technical Institute (VTI) in Žarkovo, and graphically presented in Figs. 3 and 4 in the First Paper.

Table 1 in the First Paper shows results of stress intensity factor determination using various procedures. Obviously for the adopted mesh, all of the described procedures are accurate enough.

Certain advantage should be given to the numerical calculation using the J integral and based on Eq. [16], whose programming gives us entirely automatic results.

Three-dimensional analysis of fracture mechanics problems

In every real fracture mechanics application, the stress state is usually three-dimensional. Anyhow, in most cases we can approximate it either by plane stress state or by assuming plane strain. However, this is impossible in certain practically relevant cases, such as a compact specimen. Since the specimen (Fig. 1) is neither negligibly thin (plain stress) and neither are its inner points far from the outer surfaces (plane strain), the analysis must take into account the three-dimensional stress field character. According to authors who had solved this problem /8/, the stress intensity factor calculated in symmetry plane differs by 1–2% from the theoretical plane strain value.

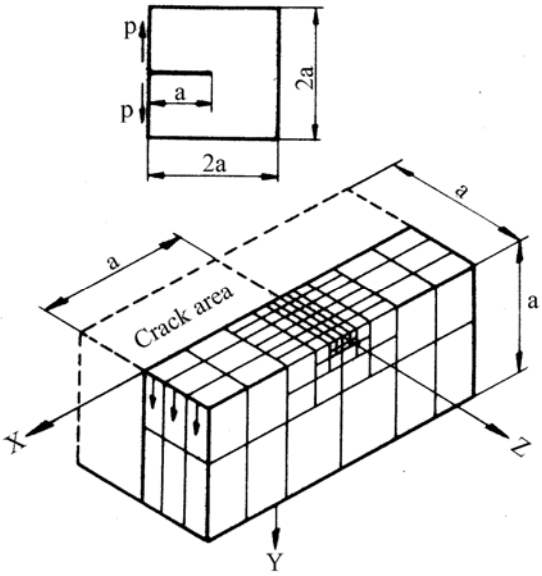

Figure 1. Compact specimen for tensile testing.

THE PROBLEM OF NUMERICAL DETERMINATION OF ELASTIC-PLASTIC BOUNDARY OF CRACKED BODY

There is no basic difference between the elastic-plastic analysis of objects with or without cracks if we use the finite element method. If we are interested in the elastic-plastic boundary, it is most usual to consider the problem as non-linear since the strains of common structural materials are very small, except in the negligible near crack tip area.

Anyhow, even such a simplified scheme comprises the solving procedure which could include non-linear problems as well. For this purpose the tangent method (Newton-Raphson) and its different modifications are used in most cases. However, the complex procedure of calculating the so called stiffness tangent matrix requires application of special programmes for non-linear problems which are currently available to a limited number of users of the finite element method.

Anyhow, it should be mentioned that programmes for linear analysis by the finite element method can be applied in a simple way on solving non-linear problems, especially materially non-linear. The secant method is used in this case (Fig. 2). The procedure is reduced to determining secant modulus from the σ–ε curve after successive iterations. With these modules we calculate new stiffness matrixes. Experience from the author of this chapter and his co-workers suggest that after 4-5 iterations, technically sufficient accuracy can be achieved. From the mathematical point of view, this convergence is linear, while for the tangent method it is square. However, the advantage lies in the fact that there is no accumulation of consecutive iteration inaccuracies and that the stiffness matrixes are generally better conditioned compared to the tangent method.

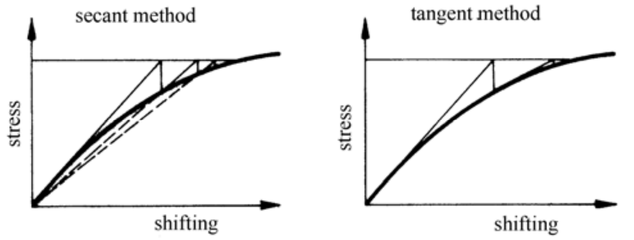

Figure 2. Methods for solving non-linear problems.

Once, after a completed iterative procedure, the iso-stress lines are drawn (see the First Paper, Fig. 3); the line that corresponds to the yield stress for the considered material is the elastic-plastic boundary.

THE PROSPECTIVES OF NUMERICAL ANALYSIS OF FRACTURE MECHANICS PROBLEMS

Some of the most difficult fracture mechanics problems, such as determining fracture mechanics parameter' critical values have not been considered in this chapter. This problem is in close connection with crack tip blunting during deformation (First Paper, Fig. 4). Because of this blunting, the crack tip stresses have finite values, and the next crack growth occurs only after stress values exceed the ultimate tensile strength for the material. This problem is, apparently, not only materially non-linear, but also geometrically non-linear. However, also in this case we can apply the iterative procedure from the previous section with modifications of geometric data after each iteration, but always starting with referent configurations, and by using non-linear strain-displacement relations. Also, there are no principle obstacles for the numerical analysis to include uncoupled and coupled thermal and dynamic fracture mechanics problems /10,11/.

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