First Paper

The original paper (in Serbian) appears in monograph "Introduction to Fracture Mechanics and Fracture-Safe Design," edited by S. Sedmak, jointly published by GOŠA Institute (Smederevska Palanka) and the Faculty of Technology and Metallurgy (Belgrade), pp. 107-124, 1980. The monograph contains lectures from the First International Fracture Mechanics Summer School (IFMASS 1) held in Smederevska Palanaka, 1980.

Originalna verzija ovog rada objavljena je na srpskom jeziku u monografiji "Uvod u mehaniku loma i konstruisanje sa sigurnošću od loma", urednik S. Sedmak, zajedničko izdanje Instituta GOŠA (Smederevska Palanka) i Tehnološko-metalurškog fakulteta (Beograd), str. 107-124, 1980. Monografija sadrži predavanja sa Prve međunarodne letnje škole mehanike loma (IFMASS 1) u Smederevskoj Palanci, 1980.

Mladen Berković

DETERMINATION OF STRESS INTENISITY FACTORS USING FINITE ELEMENT METHOD

Original scientific paper, UDC: 539.421:519.673

Abstract

The stress intensity factor is one of Fracture Mechanics' basic parameters. In this paper its determination is considered by using the finite element method (FEM).

Furthermore, effective determination of stress intensity factors by a pack of general purpose programmes in structural mechanics SMS, developed in the Aeronautical Institute (VTI) Žarkovo, will be represented.

INTRODUCTION

The most important problem in fracture mechanics is how to assess stress distribution near a sharp tip of a crack. Assuming linear elasticity, the stress on the tip is singular, therefore the fundamental parameters taken into account are stress intensity factors K_{I}, K_{II} and K_{III}, which represent a measure of intensity of stress singularity of three basic forms of crack development (opening, in-plane shearing and out-of-plane shearing), as shown in Fig. 1.

Brittle fracture occurs when loading and crack length correspond to critical value of the stress intensity factor.

In addition, crack growth under effect of cyclic loading is often expressed using a simple Paris equation /3/:

$$\frac{da}{dN} = C\left(\Delta K_{I}\right)^{m} \quad (1)$$

where N is the number of loading cycles, a–crack length, C and m experimentally obtained material constants, and ΔK_{I} represents the range within which the stress intensity factor changes under cyclic loading, in other words

$$\Delta K_{I} = K_{I\,max} - K_{I\,min} \quad (2)$$

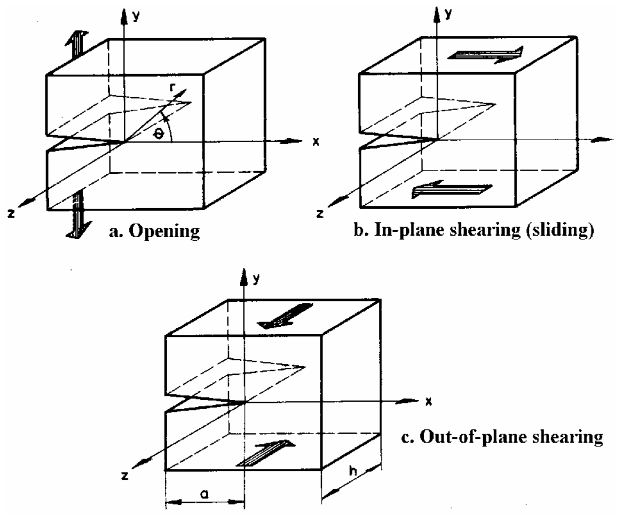

Figure 1. Three forms of crack development.

DISPLACEMENTS AND STRESSES NEAR THE CRACK TIP

Displacements and stresses around the crack tip can be represented using formulae /1,5/:

$$
\begin{aligned}
u_1 &= \frac{1}{2\mu}\sqrt{\frac{r}{2\pi}}\left\{K_{\mathrm{I}}\left[(2k-1)\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right]+K_{\mathrm{II}}\left[(2k+3)\sin\frac{\theta}{2}+\sin\frac{3\theta}{2}\right]\right\}\\
u_2 &= \frac{1}{2\mu}\sqrt{\frac{r}{2\pi}}\left\{K_{\mathrm{I}}\left[(2k+1)\sin\frac{\theta}{2}-\sin\frac{3\theta}{2}\right]+K_{\mathrm{II}}\left[(2k-3)\cos\frac{\theta}{2}+\cos\frac{3\theta}{2}\right]\right\}\\
u_3 &= \frac{1}{\mu}\sqrt{\frac{2r}{\pi}}K_{\mathrm{III}}\sin\frac{\theta}{2}
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\sigma^{11} &= \frac{1}{\sqrt{2\pi r}}\left[K_{\mathrm{I}}\cos\frac{\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)-K_{\mathrm{II}}\sin\frac{\theta}{2}\left(2+\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)\right]\\
\sigma^{12} &= \frac{1}{\sqrt{2\pi r}}\left[K_{\mathrm{I}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}+K_{\mathrm{II}}\cos\frac{3\theta}{2}\left(1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)\right]\\
\sigma^{13} &= -\frac{1}{\sqrt{2\pi r}}K_{\mathrm{III}}\sin\frac{\theta}{2}\\
\sigma^{22} &= \frac{1}{\sqrt{2\pi r}}\left[K_{\mathrm{I}}\cos\frac{\theta}{2}\left(1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)+K_{\mathrm{II}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right]\\
\sigma^{23} &= \frac{1}{\sqrt{2\pi r}}K_{\mathrm{III}}\cos\frac{\theta}{2}\\
\sigma^{33} &= \frac{3-\nu-k(1+\nu)}{\nu}\frac{1}{\sqrt{2\pi r}}\left(K_{\mathrm{I}}\cos\frac{\theta}{2}-2K_{\mathrm{II}}\sin\frac{\theta}{2}\right)
\end{aligned}
\tag{4}
$$

In the expressions above, u_1, u_2 and u_3 are the displacement components in directions x^1, x^2, x^3 parallel to the normal, binormal and tangent of the crack tip, in x^1, x^2 plane with polar coordinates r, θ relative to the penetration point of the crack tip through the plane; μ represents the shear modulus, ν is Poisson's ratio, $k = 3 - 4\nu$, except in case of plane stress, where $k = (3 - \nu)/(1 + \nu)$.

Linear elastic fracture mechanics can be applied only when the plastic strain, which always occurs since the stress singularity $r = 0$, is limited to the region near the crack tip which is much smaller than other relevant dimensions /1, 2/. One of these applications, typical for small plastic zone, is the brittle fracture of high-strength materials during crack growth in small amplitude regime and at a large number of loading cycles.

Stress intensity factors are also related to strain energy U by expressions [2], [4], [6] and [7]:

$$G = \frac{dU}{dA} = \frac{k+1}{8\mu}\left(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2\right) + \frac{1}{2\mu}K_{\mathrm{III}}^2 \tag{5}$$

and in case of plane stress:

$$G = \frac{dU}{dA} = \frac{k+1}{8\mu}\left(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2 + K_{\mathrm{III}}^2\right) \tag{6}$$

where G is the measure of released strain energy during crack growth; the element of the crack area is

$$dA = hda \tag{7}$$

with h–thickness, and a–crack width. In case of plane strain, $h = 1$.

Another possibility for calculating the measure of released strain energy, in case of a plane problem, is based on a path-independent J integral /6,12/,

$$J = \int_{\Gamma}\left(Wdx^2 - \sigma^{ij}n_j\frac{\partial u_i}{\partial x^1}ds\right) \tag{8}$$

Strain energy per unit volume is represented as

$$W = \frac{1}{2}\sigma^{ij}e_{ij} \tag{9}$$

Taking into account the symmetry of stress tensors σ^{ij}, one can write:

$$W = \frac{1}{2}\sigma^{ij}u_{ij} \tag{10}$$

where strain tensor e_{ij} is replaced by displacement gradient

$$u_{ij} = \frac{\partial u_i}{\partial x^j} \tag{11}$$

In the plane problem, normal vector coordinates are:

$$n_1 = \frac{dx^2}{ds},\ n_2 = -\frac{dx^1}{ds} \tag{12}$$

In a special case of generalized plane stress, stress tensor coordinates are:

$$\sigma^{ij} = \frac{2\mu}{1-\nu}\left[\nu\delta^{ij}\delta^{kl} + (1-\nu)\delta^{ik}\delta^{jl}\right]u_{(k,l)} \tag{13}$$

By placing [10], [12] and [13] in [8] the following expression is finally obtained:

$$J = \frac{1}{2}\frac{\mu}{1-\nu}\int_{\Gamma}\left\{\begin{bmatrix}(1-\nu)\left(u_{1,2}+u_{2,1}\right)\left(u_{1,2}-u_{2,1}\right)+\\+2\left(u_{2,2}+u_{1,1}\right)\left(u_{2,2}-u_{1,1}\right)\end{bmatrix}dx^2 + \atop +2\begin{bmatrix}(1-\nu)u_{1,1}u_{1,2}+(1+\nu)u_{1,1}u_{2,1}+\\+2u_{2,1}u_{2,2}\end{bmatrix}dx^1\right\} \tag{14}$$

DETERMINATION OF STRESS INTENSITY FACTORS USING FINITE ELEMENT METHOD

The finite element method, as well as fracture mechanics represent an area that has developed rapidly. Therefore, a large number of possibilities for calculating fracture mechanics parameters by finite element method are available. Partial review of these possibilities can be found in /4,6,8 and 9/.

The possibilities of determining fracture mechanics parameters by standard software, available to everyone who uses the finite element method, will be considered.

As it is well known, the application of this method requires forming of a corresponding mesh that divides the observed body into a number of uniform elements. In a case where standard programmes of the finite element method are used, it is necessary to refine significantly the mesh near the crack tip. If the corresponding programme, such as the SMS package has the pentagonal planar element option /10,11/, this kind of refinement is quite easy to perform.

The application of this method to an observed problem results in mesh displacement, which leads to determination of stresses. Once these quantities are known, one can determine the corresponding fracture mechanics parameters by using Eqs. [3], [4], [5], [6] and [14].

DETERMINATION OF STRESS INTENSITY FACTORS BASED ON GIVEN DISPLACEMENT VALUES

It is obvious considering Eqs. [3] that they can be solved in regard to K_I, K_{II}, and K_{III} for known coordinates r and θ and values u_1, u_2 and u_3. For practical reason, it will be adopted that $\theta = \pi$, meaning that the displacements on the crack surface are considered, so from Eqs. [3] it follows:

$$\begin{aligned} K_{\mathrm{I}} &= \frac{\mu u_2}{k+1}\sqrt{\frac{2\pi}{r}} \\ K_{\mathrm{II}} &= \frac{\mu u_1}{k+1}\sqrt{\frac{2\pi}{r}} \\ K_{\mathrm{III}} &= -\mu u_3\sqrt{\frac{\pi}{2r}} \end{aligned} \tag{15}$$

For effective determination of these values, it is usual to calculate several of them along the line θ = const, and then determine the valid one by extrapolation to $r = 0$. This procedure, simple as it is, still requires time and effort for graphic elaboration of results.

DETERMINATION OF STRESS INTENSITY FACTORS BASED ON GIVEN STRESS VALUES

This procedure is analogous to the previous one, with the use of Eqs. [4]. Again, we will determine K_I, K_{II}, K_{III} assuming $\theta = 0$:

$$\begin{aligned} K_{\mathrm{I}} &= \sigma^{22}\sqrt{2\pi r} \\ K_{\mathrm{II}} &= \sigma^{12}\sqrt{2\pi r} \\ K_{\mathrm{III}} &= \sigma^{23}\sqrt{2\pi r} \end{aligned} \tag{16}$$

This procedure is usually less accurate than the above, because the stress is less accurate compared to the displacement when using FEM.

DETERMINATION OF STRESS INTENSITY FACTORS BY USING THE MEASURE OF RELEASED STRAIN ENERGY DURING CRACK GROWTH

When using this approach, one can explicitly determine the stress intensity factors only in case of a pure form of deformation (Fig. 1). For example, if $K_{II} = K_{III} = 0$, it follows from [5] and [6]:

$$K_I = \sqrt{\frac{8\mu}{k+1}G} \tag{17}$$

Released strain energy, G, can be approximately calculated using two consecutive FEMs for two close crack lengths which differ by an increment Δa. From [5] and [6] follows:

$$G = \frac{\Delta U}{h\Delta a} \tag{18}$$

where

$$\Delta U = U_2 - U_1 \tag{19}$$

represents the difference due to two close crack lengths, exposed to the same external loading.

As for strain energy, while solving the static problems with the theory of linear elasticity using FEM, it equals:

$$U = \frac{1}{2}R^T u \tag{20}$$

where R is an n–dimensional vector designating the external forces affecting the structure, and u is the vector of their corresponding displacements. The disadvantage of this procedure is that it requires two analyses, or in performing certain corrections in the programme that are not simple /7/.

DETERMINATION OF STRESS INTENSITY FACTORS USING RICE'S J–INTEGRAL

It can be shown that J-integral equals G, therefore, from Eq. [17], one can write:

$$K_{\mathrm{I}} = \sqrt{\frac{8\mu}{k+1}J} \tag{21}$$

As for the effective determination of the J-integral, the author evaluates Eq. [14], developed in this paper, as very convenient.

On parts of the integration path that are parallel to the x_1 axis, dx_2 equals 0, and vice-versa, which reduces the amount of calculations required in [14], e.g. in case of rectangular path (Fig. 2).

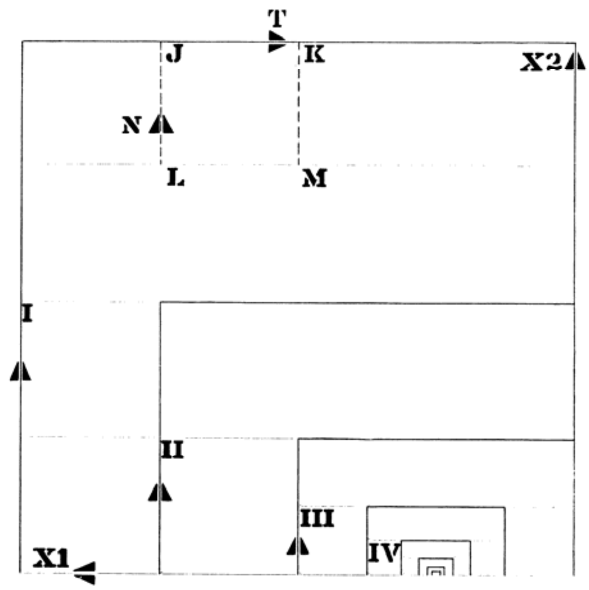

Figure 2. Effective calculation of J-integral for rectangular paths.

Derivatives in Eq. [14] can be determined from known displacement fields in an element or, if these are not known, by using the finite element method.

The derivative of displacement in the tangential direction can be calculated for the assumed linear displacement field, from:

$$\frac{\partial u}{\partial t} = \frac{u_K - u_J}{t_K - t_J} \tag{22}$$

and in the normal direction they are an average value of:

$$\frac{\partial u}{\partial n} = \frac{1}{2}\left(\frac{u_K - u_M}{n_K - n_M} + \frac{u_J - u_L}{n_J - n_L} \right) \tag{23}$$

In these expressions, t_K and n_K are coordinates of point K in Cartesian coordinate system NT. Also, in Eq. [14] dx^1 is replaced by $x^1{}_K - x^1{}_J$ and dx^2 by $x^2{}_K - x^2{}_J$. Finally, the integral in [14] is replaced with a sum of segments of the observed contour:

$$J = \frac{\mu}{1-\nu} \sum_{J,K=1,2}^{j<k=N-1,N} \left\{ \begin{array}{l} \left[(1-\nu)\left(u_{1,2}+u_{2,1}\right)\left(u_{1,2}-u_{2,1}\right) + 2\left(u_{2,2}+u_{1,1}\right)\left(u_{2,2}-u_{1,1}\right)\right]\left(x_K^2 - x_J^2\right) + \\ 2\left[(1-\nu)u_{1,1}u_{2,2} + (1+\nu)u_{1,1}u_{2,1} + 2u_{2,1}u_{2,2}\right]\left(x_K^1 - x_J^1\right) \end{array} \right\} \tag{24}$$

EXAMPLE

A square plate with a central symmetrical crack has been considered as an example. The used mesh is shown in Fig. 3.

It is automatically generated by a special pre-processor for programme package SMS. The same figure shows very good agreement of stress results with some known solutions.

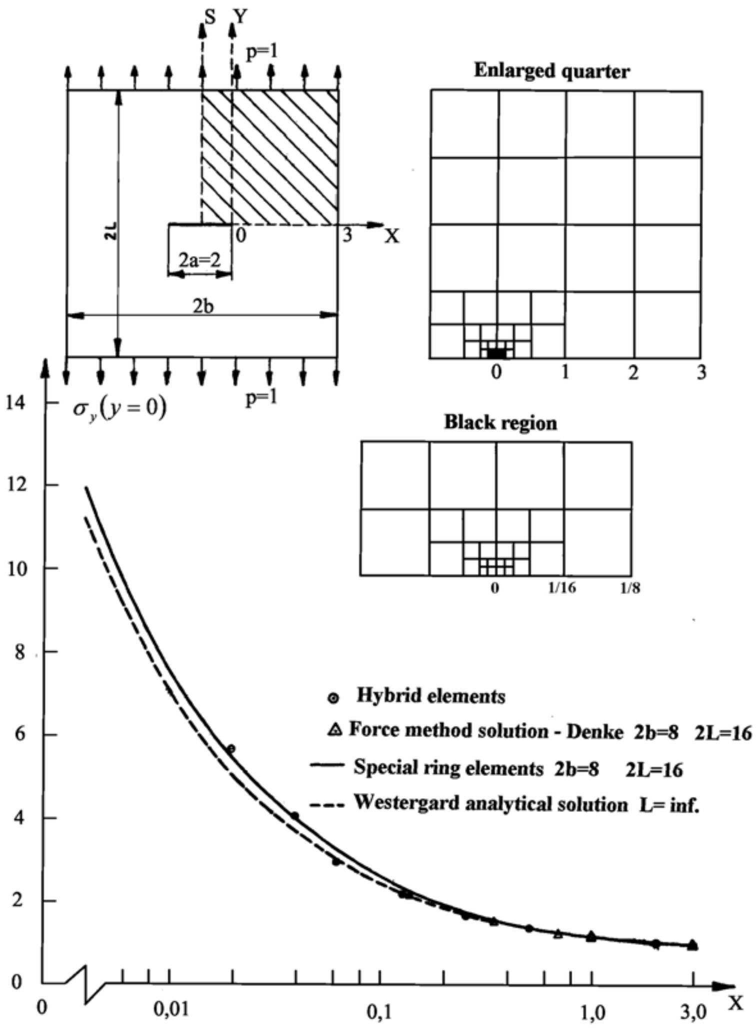

Figure 3. Square plate with a central symmetrical crack.

Further, Fig. 4 shows the lines of local stress values near the crack, calculated as Mises stresses:

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2}$$

where σ_1 and σ_2 represent principle stresses.

Figure 5 shows a deformed configuration of the considered plate, in an exaggerated scale. Graphs in Figs. 2, 4 and 6 are drawn using a standard postprocessor for programme package SMS, on a GERBER plotter.

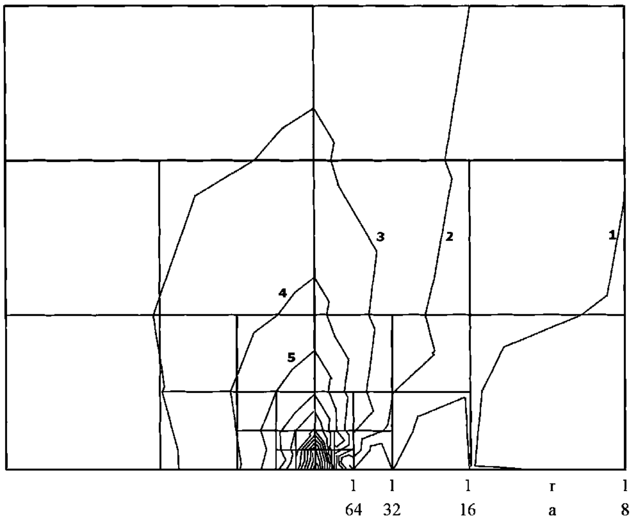

Figure 4. Local iso-stress lines.

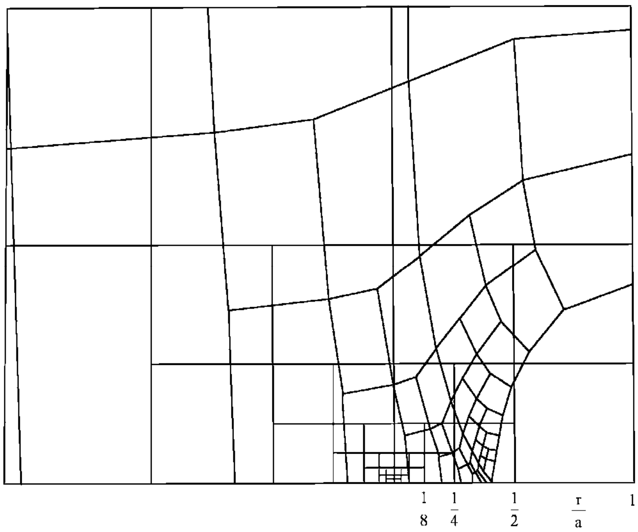

Figure 5. Deformed structure.

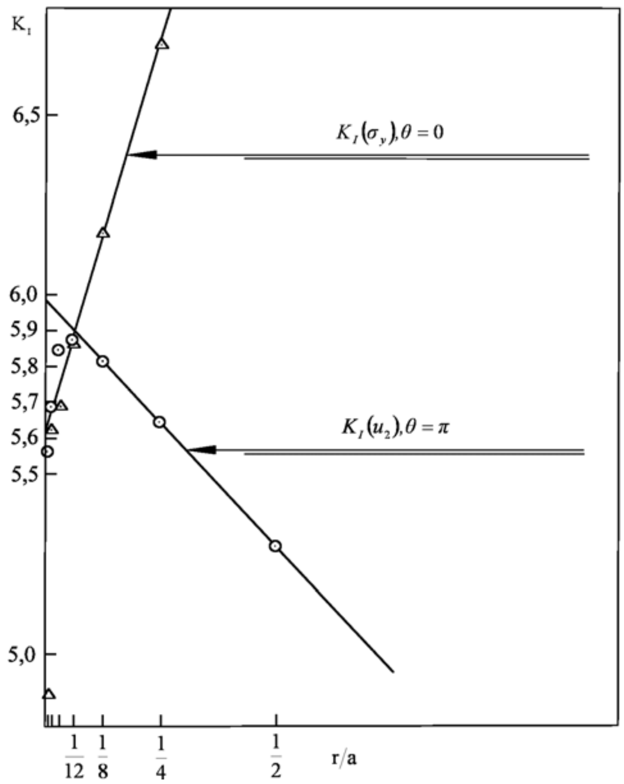

Figure 6. Graphic determination of stress intensity factors based on displacement $K_I(u_2)$ and stress $K_I(\sigma_y)$.

Figure 6 shows graphic determination of stress intensity factors, based on displacements, shown in Fig. 5, and based on stresses shown in Fig. 3, calculated using standard SMS.

Some advantage should be given to the J-integral method, because it requires a simple post-processor, based on Eq. [24] and on the different scheme previously described. Hence, the results are obtained automatically, and are a bit greater, therefore safer, compared to other procedures.

In Table 1 the results of stress intensity factor determination using various procedures are presented. Apparently, with developed mesh, all procedures are sufficiently accurate.

Table 1. Stress intensity factors $K_I = \sigma(\pi a)^{1/2} F(a/b)$.

		Stress intensity factor	Correction factor	Correction factor	Crack length to plate width ratio	Plate length to width ratio
		K_I	$F(a/b)$	$F(a/b,L)$	a/b	L/b
Reference /7/	Infinite plate	5.60	1	1	0	-
	Infinite band (ISIDA)	5.77	1.03	-	0.25	∞
	Infinite band (PARIS+SIH)	5.82	1.04	-	0.25	∞
	Finite plate (HELLEN)	6.05		1.08	0.25	1.0
This paper	Displacements	5.98		1.07	0.25	1
	Stresses	5.60		1.00		
	G – measure	5.99		1.07		
	J (I)	6.15		1.10		
	J (II)	6.14		1.10		
	J (III)	6.10		1.09		
	J (IV)	6.22		1.11		
	J (V)	6.20		1.11		
	J (VI)	6.17		1.10		
	J (VII)	6.15		1.10		
	J (VIII)	6.15		1.10		

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