Pankaj Thakur[*], *Priya Gulial*

INVESTIGATING THE INFLUENCE OF NON-UNIFORM DENSITY IN PVC, POLYSTYRENE, AND NATURAL RUBBER DISCS UNDER ROTATION

ISTRAŽIVANJE UTICAJA NEUNIFORMNE GUSTINE PRI ROTACIJI DISKOVA OD PVC, POLISTIRENA I PRIRODNE GUME

Originalni naučni rad / Original scientific paper
Rad primljen / Paper received: 1.10.2024
https://doi.org/10.69644/ivk-2026-01-0061

Adresa autora / Author's address:
Department of Mathematics, ICFAI University Himachal Pradesh, India P. Thakur https://orcid.org/0000-0001-8119-2697 ;
P. Gulial https://orcid.org/0009-0009-0266-2298
*email: pankaj_thakur15@yahoo.co.in

Keywords
- density
- disc
- yielding
- angular speed
- rubber

Ključne reči
- gustina
- disk
- tečenje
- ugaona brzina
- guma

Abstract

This study investigates the impact of non-uniform density on the mechanical performance of rotating discs made from polyvinyl chloride (PVC), polystyrene (PS), and natural rubber (NR). Traditional analyses often assume uniform density which can lead to inaccurate predictions in real-world applications. By employing a variable density model, this research examines how density variations affect stress and strain distributions during rotation. Key findings reveal that polystyrene discs exhibit the highest radial stress, while natural rubber demonstrates superior performance under radial loading. The results underscore the importance of considering non-uniform density in the design and optimisation of polymer-based rotating systems, ultimately aiming to enhance their reliability and efficiency in various engineering applications.

Izvod

Istraživanje obuhvata uticaj neuniformne gustine na mehaničke karakteristike rotirajućih diskova od polivinil hlorida (PVC), polistirena (PS) i od prirodne gume (NR). U tradicionalnim analizama se često pretpostavlja uniformna gustina, što može dovesti do netačnih predviđanja u stvarnim primenama u praksi. Uvođenjem modela sa promenljivom gustinom, ovde se istražuje kako promena gustine utiče na raspodele napona i deformacija pri rotaciji. Ključni rezultati pokazuju da polistirenski diskovi pokazuju najveći radijalni napon, dok se kod prirodne gume ispoljavaju superiorne performanse pri radijalnom opterećenju. Rezultati potvrđuju bitnost u razmatranju neuniformne gustine u projektovanju i optimizaciji polimernih rotirajućih sistema, sa ciljem da se poboljšaju njihova pouzdanost i efikasnost u raznim inženjerskim primenama.

INTRODUCTION

The study of polymer discs is essential in various engineering applications, particularly in sectors such as automotive, aerospace, and machinery. Among these materials, polyvinyl chloride (PVC), polystyrene (PS), and natural rubber are frequently utilised due to their advantageous mechanical properties, including flexibility, durability, and resistance to wear. The performance of these discs, especially under rotational conditions, is critical for ensuring the reliability and efficiency of the systems they support. However, a common assumption in traditional analyses is that these polymers exhibit uniform density throughout their structure. This oversimplification may lead to inaccurate predictions of their mechanical behaviour, as real-world conditions often reveal significant variations in density. Non-uniform density in polymer materials can arise from multiple factors, including inconsistencies in the manufacturing process, variations in material formulation, and changes in environmental conditions such as temperature and humidity. For instance, the absorption of moisture can alter the density of natural rubber, while the polymerisation process for PVC and PS can result in density gradients. These variations can significantly influence the stress and strain distributions within rotating discs, potentially affecting their structural integrity and performance. Solutions for slender isotropic discs are typically available in standard elasticity and plasticity textbooks /1-3/. Chakrabarty /4/ and Heyman /5/ addressed the plastic state by extending elastic solutions and incorporating Tresca's yield condition to define the plastic range. They further aligned elastic and plastic stresses at radius $r = c$ to derive elastic-plastic stress profiles. Perfect elasticity and ideal plasticity represent extreme material properties and applying ad-hoc rules such as the yield condition attempts to delineate these extremes with a sharp boundary, which is theoretically challenging. Seth's transition theory /6/ dispenses with assumptions such as yield criteria, incompressibility conditions, and associated flow rules, thereby addressing a broader problem encompassing specific cases derived from these assumptions. The theory in /6/ employs generalised strain measures and asymptotic solutions at critical or turning points of the differential equations defining the deformed field, demonstrating successful applications across a wide range of problems /7-18/.

OBJECTIVE OF THE STUDY

The objective of this research is to investigate how non-uniform density affects the mechanical performance of rotating discs made from PVC, polystyrene, and natural rubber. By analysing variations in density, the study aims to understand their impact on stress and strain distributions during operation. This research seeks to develop a more accurate model of material behaviour under rotational loads, enhancing the design and optimisation of polymer-based systems. Ultimately, the goal is to provide insights that improve the reliability and efficiency of these discs in practical engineering applications, contributing to safer and more effective solutions across various industries.

GOVERNING EQUATIONS

We examine a thin annular disc with variable density that has a central bore defined by an inner a, and outer radius b. The disc rotates around an axis that is perpendicular to its plane and goes through its centre, with an angular speed that gradually increases. The disc's thickness is considered minimal; allowing us to assume it is in a state of plane stress, which means the axial stress is effectively zero. The equation of stress is given by Thakur, /8/:

$$T_{rr}=\frac{2\mu}{n}\left[3-2C-\beta^n\left\{1-C+(2-C)\left(\frac{r\beta'}{\beta}+1\right)^n\right\}\right],$$

$$T_{\theta\theta}=\frac{2\mu}{n}\left[3-2C-\beta^n\left\{2-C+(1-C)\left(\frac{r\beta'}{\beta}+1\right)^n\right\}\right],$$

$$T_{r\theta}=T_{\theta z}=T_{zr}=T_{zz}=0\,. \tag{1}$$

Equations of equilibrium are all satisfied except

$$T_{\theta\theta}-\frac{d}{dr}(rT_{rr})=\rho\omega^2r^2\,. \tag{2}$$

Inserting Eq.(1) into Eq.(2), we get a nonlinear differential equation in β as

$$(2-C)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta}=\frac{n\rho\omega^2r^2}{2\mu}+\beta^n\times$$

$$\times\left[1-(P+1)^n-nP\{1-C+(2-C)(P+1)^n\}\right]. \tag{3}$$

The transitions of β in Eq.(3) are: $P \to -1$ and $P \to \pm\infty$. The boundary conditions are:

$$T_{rr}=0 \quad \text{at} \quad r=a \quad \text{and} \quad r=b\,. \tag{4}$$

SOLUTION

It has been shown that the asymptotic solution through the principal stress leads from elastic to plastic state /7-18/ at the transition point $P \to \pm\infty$. The transition function R is defined as:

$$R=\frac{n}{2\mu}T_{\theta\theta}=\left[(3-2C)-\beta^n\{2-C+(1-C)(P+1)^n\}\right]. \tag{5}$$

Taking the logarithmic differentiating of Eq.(5) w.r.t. r and substituting it into Eq.(2), taking asymptotic value $P \to \pm\infty$, after integration we get:

$$R=A_1r^{\nu-1}\,. \tag{6}$$

From Eqs.(5) and (6), it follows:

$$T_{\theta\theta}=\left(\frac{2\mu}{n}\right)A_1r^{\nu-1}\,. \tag{7}$$

By substituting Eq.(7) into Eq.(2) and using density parameter $\rho=\rho_0(r/b)^{-m}$ then integrating, we get:

$$T_{rr}=\frac{B_1}{r}+\frac{2\mu A_1r^{\nu-1}}{n\nu}-\frac{\rho_0\omega^2b^mr^{2-m}}{(3-m)}\,,\quad m\neq 3. \tag{8}$$

Using boundary condition Eq.(4) into Eq.(8), we get

$$A_1=\frac{\rho_0\omega^2n\nu b^m(b^{3-m}-a^{3-m})}{2\mu(3-m)(b^\nu-a^\nu)}\,,$$

$$B_1=\frac{\rho_0\omega^2b^ma^{3-m}}{(3-m)}-\frac{\rho_0\omega^2b^m(b^{3-m}-a^{3-m})}{(3-m)(b^\nu-a^\nu)}a^\nu,\quad m\neq 3. \tag{9}$$

By substituting Eq.(9) into Eqs.(7) and (8), we get:

$$\tau_r=\frac{\Omega_i^2}{(3-m)R}\left[\frac{1-R_0^{3-m}}{1-R_0^\nu}(R^\nu-R_0^\nu)-R^{3-m}+R_0^{3-m}\right],$$

$$\tau_\theta=\frac{\Omega_i^2\nu}{(3-m)R}\left[\frac{1-R_0^{3-m}}{1-R_0^\nu}\right]R^\nu \quad \text{for} \quad m\neq 3\,, \tag{10}$$

where: Y is yielding stress; m is density parameter; ν is Poisson's ratio; and $\Omega_i^2=\rho_0\omega^2b^2/Y$, $R=r/b$, $R_0=a/b$, $\tau_{rr}=T_{rr}/Y$, $\tau_{\theta\theta}=T_{\theta\theta}/Y$ are non-dimensional components. Equation (10) gives elastic-plastic transitional stresses in a thin rotating disc of variable thickness with edge loading. Now,

$$\tau_r-\tau_\theta=\frac{\Omega_i^2}{(3-m)R}\left[\frac{1-R_0^{3-m}}{1-R_0^\nu}(R^\nu-R_0^\nu-\nu R^\nu)-R^{3-m}+R_0^{3-m}\right],$$

$$m\neq 3\,. \tag{11}$$

Equation (11) indicates that the difference $|\tau_r-\tau_\theta|$ reaches its maximum at the inner surface (where $r=a$). Consequently, yielding will occur at this inner surface, leading to the modified form of Eq.(11): $|\tau_r-\tau_\theta|_{R-R0}=1$, then we find:

$$\Omega_i^2\Big|_{\text{intital yielding}}=\frac{(3-m)(1-R_0^\nu)}{(1-R_0^{3-m})\nu}R_0^{1-\nu},\ m\neq 3\,. \tag{12}$$

Stresses for initial yielding surface: by substituting Eq.(12) into Eq.(11), we have:

$$\tau_r\Big|_{\text{initial yielding}}=\frac{(1-R_0^\nu)R_0^{1-\nu}}{(1-R_0^{3-m})R\nu}\left[\frac{1-R_0^{3-m}}{1-R_0^\nu}(R^\nu-R_0^\nu)-R^{3-m}+R_0^{3-m}\right],$$

$$\tau_\theta\Big|_{\text{initial yielding}}=R_0^{1-\nu}R^{\nu-1} \quad \text{for} \quad m\neq 3\,. \tag{13}$$

For angular speed $\omega_f>\omega_i$, at which the rotating disc reaches a fully plastic state, from Eq.(11) we derive at the outer surface:

$$\Omega_f^2\Big|_{\text{fully-plastic state}}=\frac{2(3-m)(1-\sqrt{R_0})}{(1-R_0^{3-m})}\,. \tag{14}$$

Stresses in the fully plastic state: by substituting Eq.(14) in Eq.(10), we obtain the expressions for the fully plastic state:

$$\tau_r\Big|_{\text{fully-plastic state}}=\frac{2(1-\sqrt{R_0})}{R(1-R_0^{3-m})}\left[\frac{(1-R_0^{3-m})}{(1-\sqrt{R_0})}(\sqrt{R}-\sqrt{R_0})-\right.$$

$$\left.-R^{3-m}+R_0^{3-m}\right],$$

$$\tau_\theta\Big|_{\text{fully-plastic state}}=\frac{1}{\sqrt{R}}\,. \tag{15}$$

VALIDATION OF RESULTS

By neglecting the density parameter from the results given by Thakur /8/, the presented results Eqs.(12)-(15) are the same.

RESULTS AND DISCUSSION

To compute the stresses and angular speed from the given analysis, the following values are used: Poisson's ratio is set at $\nu = 0.33$ for PS, $\nu = 0.38$ for PVC, and $\nu = 0.4999$ for NR. The parameters chosen include an inner radius $a = 1$ mm, an outer radius $b = 2$ mm, and a reference radius $R_0 = 0.5$. In Figs. 1 and 2, curves are presented showing the relationship between radial and circumferential stresses versus R for both the initial yielding and fully plastic states. Specifically, in Fig. 1, it is clear that the radial stress (excluding the inner and outer radii of the disc) is the highest for the disc made of polystyrene compared to polyvinyl chloride and natural rubber. Additionally, the radial stress decreases with increasing density parameter m = -0.5 and m = 0.5, in respect.

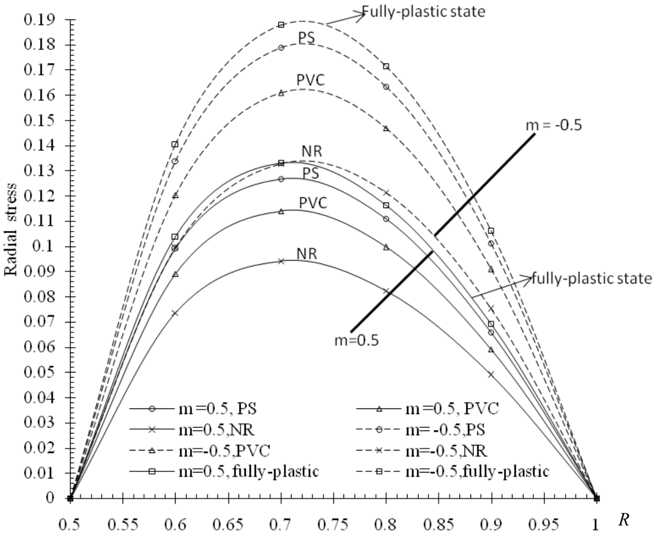

Figure 1. Radial stress vs. *R*.

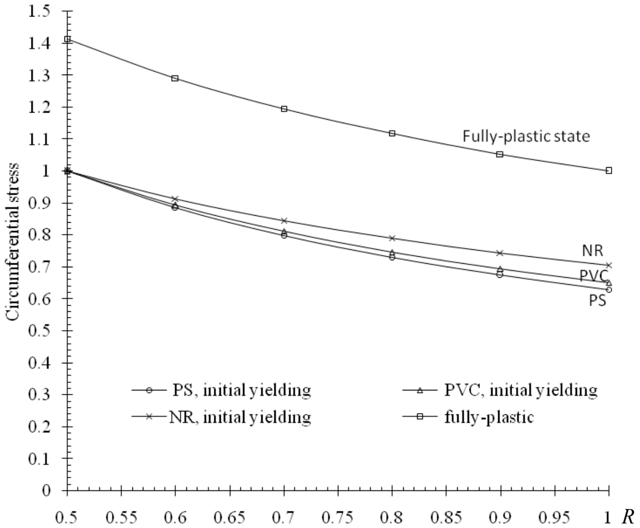

Figure 2. Circumferential stress vs *R*.

In Fig. 2, it is observed that circumferential stresses are the highest at the inner surface. For the outer surfaces, the disc made of NR requires greater radial stress compared to those made of PS and PVC. Therefore, natural rubber demonstrates better performance than both polystyrene and polyvinyl chloride in this context.

CONCLUSIONS

The main findings are:

- radial stress is highest in PS discs compared to PVC and NR;
- radial stress decreases with increasing density parameters;
- circumferential stresses peak at the inner surface;
- Natural rubber requires greater radial stress at the outer surface than PS and PVC;
- overall, NR exhibits superior performance compared to both PS and PVC.

ACKNOWLEDGEMENTS

This research article is especially dedicated to Prof. Dr Aleksandar Sedmak (editor-in-chief Emeritus), Prof. Dr Zoran Radaković (editor-in-chief) of *Structural Integrity and Life* journal and the Society for Structural Integrity and Life 'Prof. Dr. Stojan Sedmak' and the IMS Institute, Serbia.

REFERENCES

1. Sokolnikoff, I.S., Mathematical Theory of Elasticity, 2nd Ed., McGraw Hill Book Co., New York, 1956.
2. Timoshenko, S.P., Goodier, J.N., Theory of Elasticity, 3rd Ed., McGraw-Hill Book Co., New York, USA, 1970.
3. Blazynski, T.Z., Applied Elasto-Plasticity of Solids, McMillan Press Ltd., London, 1983.
4. Chakrabarty, J., Theory of Plasticity, McGraw-Hill, New York, 1987.
5. Heyman J., (1958), *Plastic design of rotating discs*, Proc. Inst. Mech. Eng. 172(1): 531-547. doi: 10.1243/PIME_PROC_1958_172_045
6. Seth, B.R. (1962), *Transition theory of elastic-plastic deformation, creep and relaxation*, Nature, 195: 896-897. doi: 10.1038/195896a0
7. Seth, B.R. (1966), *Measure concept in mechanics*, Int. J Non-Linear Mech. 1(1): 35-40. doi: 10.1016/0020-7462(66)90016-3
8. Thakur, P., *Some problems in elastic-plastic and creep transition*, Ph.D. Thesis, Dept. of Mathematics and Statistics, Himachal Pradesh University, Shimla, India, 2006 https://shodhganga.inflibnet.ac.in/handle/10603/121294
9. Thakur, P. (2013), *Creep transition stresses of orthotropic thick-walled cylinder under combined axial load under internal pressure*, Facta Universities Ser.: Mech. Eng. 11(1): 13-18.
10. Temesgen, A.G., Singh, S.B., Thakur, P. (2020), *Modeling of creep deformation of a transversely isotropic rotating disc with a shaft having variable density and subjected to a thermal gradient*, Therm. Sci. Eng. Prog. 20 (100745). doi: 10.1016/j.tsep.2020.100745
11. Thakur P., Sethi, M., Gupta, K., Bhardwaj R.K. (2021), *Thermal stress analysis in a hemispherical shell made of transversely isotropic materials under pressure and thermo-mechanical loads*, Zeitschrift für angewandte Mathematik und Mechanik, 101(12): e202100208. doi: 10.1002/zamm.202100208
12. Thakur, P., Sethi, M., Kumar, N., et al. (2021), *Analytical solution of hyperbolic deformable disk having variable density*, Mech. Solids, 56(6): 1039-1046. doi: 10.3103/S0025654421060194
13. Chand, S., Sood, S, Thakur, P., Gupta, K. (2023), *Elasto-plastic stress deformation in an annular disk made of isotropic material and subjected to uniform pressure*, Struct. Integr. Life, 23 (1): 61-64.
14. Singh, N., Kaur, J., Thakur, P., Murali, G. (2023), *Structural behaviour of annular isotropic disk made of steel/copper material with gradually varying thickness subjected to internal pressure*, Struct. Integr. Life, 23(3): 293-297.

15. Sukhvinder, Gulial, P., Pathania, D.S., et al. (2024), *Comparative study of creep in a disk made of rubber/copper material and fitted with rigid shaft*, Struct. Integr. Life, 24(2): 159-166. doi: 10.69644/ivk-2024-02-0159
16. Kumar, S., Thakur, P., Sood, S., et al. (2024), *Transversely isotropic elastoplastic behaviour in a mechanically loaded rotating disk*, Struct. Integr. Life, 24(2): 167-171. doi: 10.69644/ivk-2024-02-0167
17. Sukhvinder, Gulial, P., Pathania, D.S., et al. (2024), *Thermal stress distribution in a tube of natural rubber/polyurethane material and subjected to internal pressure and mechanical load*, Struct. Integr. Life, 24(2): 151-158. doi: 10.69644/ivk-2024-02-0151
18. Singh, A., Gulial, P., Thakur, P. (2024), *Exploring the effective stress behavior of internally pressurized cylinders with varying density*, ZAMM - J Appl. Math. Mech. 104(8): e202400254. doi: 10.1002/zamm.202400254