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CASE-WISE STUDY OF TORSIONAL WAVE PROPAGATION IN A TWO-LAYERED HETEROGENEOUS MEDIA

STUDIJA SLUČAJA PROSTIRANJA TORZIONOG TALASA U DVOSLOJNOJ HETEROGENOJ SREDINI

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- jednačina disperzije

Abstract

The present article investigates the propagation of torsional waves through a two-layer structured inhomogeneous elastic medium, where material parameters rigidity and depth parameter z change, and so does density. A case-wise study is conducted for the upper inhomogeneous layer. In the first case, the variation of parameters is governed by a power-law model, while in the second case, it follows the outcome of a polynomial power and an exponential term. For the space in the lower half, the variation is modelled using the result of multiplying an exponential function by a linear function. A mathematical model is formulated based on the elastic theory of wave propagation. The technique for separating variables is employed in both layers to obtain the displacement fields. Applying intrinsic boundary conditions yields a closed-form expression regarding the frequency equation. The study reveals a significant influence of density and stiffness profiles on the torque-related phase velocity of torsional waves. Visual illustrations are provided to validate and support the observations obtained from the relationship of dispersion. The study may be helpful in geophysical exploration, seismic interpretation, and in understating wave propagation in complex surface structures.

The analysis exhibits that the presence of inhomogeneity in the medium greatly affects the propagation behaviour of torsional waves. It is observed that variations in the elastic parameters and density distribution have a noticeable impact on the phase velocity. The dispersion relation governing the propagation of torsional waves is derived in a compact form by evaluating the determinant of the coefficient matrix. The result obtained in the present work provides a theoretical framework for understanding the propagation of torsional wave in a layered heterogeneous geological structure.

Izvod

U ovom radu istražujemo prostiranje torzionih talasa kroz dvoslojnu nehomogenu elastičnu sredinu, sa promenljivim parametrima krutosti i gustine sa dubinom, Urađena je studija slučaja za ovaj nehomogeni sloj. U prvom slučaju, promena parametara je definisana po modelu stepenog zakona, dok se u drugom slučaju promena definiše zakonima stepenog polinoma i eksponencijalnog člana. U donjem poluprostoru se promene modeliraju upotrebom rezultata množenja eksponencijalne funkcije linearnom funkcijom. Matematički model se formuliše na osnovu teorije elastičnosti prostiranja talasa. Uvodi se postupak za razdvajanje promenljivih u oba sloja radi dobijanja polja pomeranja. Primenom unutrašnjih graničnih uslova dobija se izraz zatvorenog tipa s obzirom na jednačinu frekvencije. U radu se otkriva značajan uticaj profila gustine i krutosti na faznu brzinu torzionih talasa, koja je vezana sa obrtnim momentom. Date su grafičke ilustracije za opisivanje i proveru posmatranja dobijenih izrazom za disperziju. Rad može poslužiti u geofizičkim istraživanjima, interpretacijama seizmičkih pojava, i u razumevanju prostiranja talasa u složenim površinskim strukturama.

Analiza pokazuje da prisustvo nehomogenosti sredine u velikoj meri utiče na ponašanje prostiranja torzionih talasa. Primećuje se da promene u elastičnim parametrima i u raspodeli gustine imaju vidljivog uticaja na faznu brzinu. Jednačina disperzije kojom se opisuje prostiranje torzionih talasa je izvedena u kompaktnom obliku, i to procenom determinante matrice koeficijenata. Dobijeni rezultati pružaju teoretski okvir za razumevanje prostiranja torzionih talasa u slojevitoj heterogenoj geološkoj strukturi.

INTRODUCTION

Over the previous several years, earthquakes have caused devastating disasters on Earth and have strongly drawn the attention of mankind. Considering a realistic Earth model, investigation of surface waves in layered media with different elastic characteristics and heterogeneities has garnered significant scientific interest and opened avenues for numerous technological applications. Many mathematicians and additionally seismologists have followed the proposed basic theory of Love /1/ who offered an in-depth analysis of seismic wave propagation, highlighting its unique features in contrast to the isotropic media in heterogeneous anisotropic media.

A material is said to be anisotropic if its properties vary with direction, meaning its mechanical response depends upon the orientation of the applied force. Unlike isotropic materials, anisotropic materials require more than two independent elastic constants to describe their behaviour. The elasticity tensor C_{ijkl} can possess up to 81 independent components. Such materials are commonly used to model crystals, composites and layered geological media. The study of wave phenomena in anisotropic layered crusts has garnered considerable scientific interest. A particular analysis has been done on the movement of torsional waves in an inhomogeneous half-space by Chattopadhyay et al. /9/.

A torsional wave is a horizontally polarised seismic wave characterised by a twisting motion about a vertical axis. It is a type of surface wave that exhibits distinctive features during propagation. It is well known that the Earth has diverse geological characteristics and Bullen /2/ suggested the law of density that describes how density varies with depth within the Earth's interior. This law provides valuable insights into the interrelation between pressure, density, and seismic velocities beneath the Earth's surface. In layered geological systems, torsional waves are inherently dispersive, meaning their phase velocity varies with frequency.

It is widely recognised that the Earth's crust comprises a variety of heterogeneous geological properties. Bullen /2/ showed that the Earth's interior density fluctuates with depth and changes at different speeds across distinct strata. It has been crucial to study surface wave propagation stratified media in the fields of applied mathematics, geophysics, and seismology. Given that the Earth can be modelled as a prestressed, layered and heterogeneous medium. The analysis of wave propagation in such media plays a significant role in understanding seismic wave propagation. Substantial preexisting stress may arise due to numerous influences such as thermal fluctuations, atmospheric loading, gravitational effects, long-term deformation (creep), and the pressure imposed by overburden materials.

Significant advancements in the study of surface wave propagation through heterogeneous geological media have been contributed by Vardoulakis /10/, Meissner /11/, Ewing and Jaretzky /3/, Vishwakarma et al. /12-14/, Panigrahi et al. /15/ and Maity et al. /16/. Building upon the foundational theories laid out in these seminal works, as well as the pioneering framework proposed by Biot /7/, numerous scholars have investigated the dynamics of torsional surface wave propagation in stratified Earth crust models overlying a semi-infinite mantle. Dey and Dutta /4/ have thoroughly investigated the mechanical behaviour of torsional waves in a cylindrical construction that was initially strained. Additionally, comprehensive analyses of elastic wave propagation, along with their far-reaching implications in seismological model and interpretation, have been eloquently presented in the works of Pujol /5/ and Chapman /6/. These contributions collectively enrich the theoretical and applied understanding of wave phenomena in complex geophysical settings and graphical representations.

Torsional wave propagation in a two-layer heterogeneous material is investigated in this work. Two cases are considered. In Case I, a power-law model describes how the density and stiffness change, while in Case II the impact of the inhomogeneity factors on the phase velocity of torsional waves has been examined through numerical analysis.

The numerical investigation is carried out by solving the dispersion relation for various values of the inconsistency parameters. The behaviour of non-dimensional phase velocity c^2/c_0^2 is examined by plotting it against the corresponding non-dimensional wave number kH.

The effect of each homogeneity parameter on the phase velocity and its reflection and refraction dependency on the propagation angle are shown graphically. The results could help researchers and dermatologists better comprehend body wave reflection and refraction phenomena, which would further our understanding of the interior structure of the Earth. In recent years, considerable attention has been devoted to reflection and refraction problems in elastic media.

The outcome of this study may be useful in geotechnical engineering, earthquake-resistant design and seismic exploration. When inhomogeneous material properties such as different stiffness and density are considered, wave propagation modes become more realistic.

Literature review

The study /1/ shows the theoretical framework for wave motion in elastic solids, which has been widely used in subsequent analyses. Further studies extended the analysis to layered media, particularly a single layer over a half-space, where discontinuities in material properties give rise to dispersion of torsional wave. The continuity of displacement and stress at the interface leads to characteristic dispersion relations that govern wave propagation. Investigations by Ewing and Jaretzky /3/ demonstrated the influence of stratification on wave characteristics and highlighted the importance of layered structures in seismic wave analysis. The presence of multiple interfaces results in the generation of various propagation modes and cut-off frequencies. Analytical methods such as the Thomson-Haskell matrix approach have been widely employed to obtain dispersion relations in such systems, /18/.

More realistic models consider continuously heterogeneous media in which material properties vary smoothly with depth. In such cases governing equations become variable coefficient differential equations, and their solutions are often expressed in terms of special functions such as Bessel and Hypergeometric functions /14/. Additionally, studies on anisotropic and viscoelastic media have been carried out to account for realistic material behaviour. In anisotropic media, wave velocity depends on direction, while in viscoelastic media, attenuation effects lead to decay in wave amplitude with distance.

Torsional waves in cylindrical geometries such as rods and layered cylindrical shells, have also been widely investigated due to their engineering applications. In such cases, the governing equations reduce to Bessel-type equations, and the solutions depend strongly on boundary conditions imposed at the inner and outer surfaces. These models are useful in analysing wave propagation in pipelines, boreholes and structural components, /17/.

GEOMETRY OF THE PROBLEM

An inhomogeneous layer covering an inhomogeneous half-space is used to examine how torsional surface waves behave. A cylindrical coordinate system is adopted, with the half-space being the direction of the positive z-axis. H represents the thickness of the heterogeneous anisotropic layer and the origin is located at the interface between the two layers, as illustrated in Fig. 1. The variation of inhomogeneity within both the layer as well as the half-space is considered as follows.

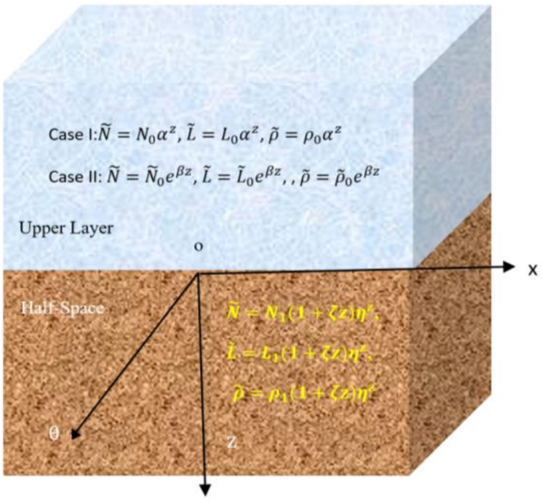

Figure 1. Problem of torsional wave geometry.

Table 1. Inhomogeneity parameters for both layers (α, ζ, η and β).

a	Case I	Case II	Half-space
$\tilde{N}$	$N_0\alpha^z$	$\tilde{N}_0 e^{\beta z}$	$N_1(1+\zeta z)\eta^z$
$\tilde{L}$	$L_0\alpha^z$	$\tilde{L}_0 e^{\beta z}$	$L_1(1+\zeta z)\eta^z$
$\tilde{\rho}$	$\rho_0\alpha^z$	$\tilde{\rho}_0 e^{\beta z}$	$\rho_1(1+\zeta z)\eta^z$

Here α, ζ, η and β are real numbers.

Derivation of displacement for the layers

The following equation has been introduced by Biot /7/

$$\frac{\partial \tilde{S}_{r\theta}}{\partial r} + \frac{\partial \tilde{S}_{z\theta}}{\partial z} + \frac{2}{r}\tilde{S}_{r\theta} = \tilde{\rho}\frac{\partial^2 \tilde{v}}{\partial t^2}, \tag{1}$$

where: $\tilde{v}$ is displacement; $\tilde{\rho}$ is density; r is radial coordinate; and θ is the circumferential coordinate.

The following equation is known as stress-strain equation,

$$\tilde{S}_{r\theta} = \tilde{N}\left(\frac{\partial \tilde{v}}{\partial r} - \frac{\tilde{v}}{r}\right),\quad \tilde{S}_{z\theta} = \tilde{L}\frac{\partial \tilde{v}}{\partial z}, \tag{2}$$

where: $\tilde{N}$, $\tilde{L}$ are directional rigidities.

Putting Eq.(2) into Eq.(1),

$$\tilde{N}\left(\frac{\partial^2 \tilde{v}}{\partial r^2} - \frac{\tilde{v}}{r^2} + \frac{1}{r}\frac{\partial \tilde{v}}{\partial r}\right) + \frac{\partial}{\partial z}\left(\tilde{L}\frac{\partial \tilde{v}}{\partial z}\right) = \tilde{\rho}\frac{\partial^2 \tilde{v}}{\partial t^2}. \tag{3}$$

The solution of Eq.(3) can be written as

$$\tilde{v} = \tilde{v}_j(r,z,t) = \phi_j(z)J_1(kr)e^{i\omega t},\quad j=1,2, \tag{4}$$

where: $J_1(kr)$ is Bessel function; k is wave number; and c is velocity ($\omega = kc$).

Using Eq.(4) in Eq.(3), we have

$$\frac{d^2\phi_j}{dz^2} + \frac{1}{\tilde{L}}\frac{d\tilde{L}}{dz}\frac{d\phi_j}{dz} - \frac{k^2\tilde{N}}{\tilde{L}}\left(1 - \frac{c^2\tilde{\rho}}{\tilde{N}}\right)\phi_j = 0,\quad j=1,2. \tag{5}$$

Displacement of the upper layer (Case-I)

On using the inhomogeneities from Case-I (Table 1) in Eq.(5), we get

$$\frac{d^2\phi_1}{dz^2} + \log\alpha\frac{d\phi_1}{dz} - E_1\phi_1 = 0, \tag{6}$$

where: $E_1 = \frac{k^2 N_0}{L_0}\left(1 - \frac{c^2}{c_0^2}\right)$; and $c_0^2 = N_0/\rho_0$.

The solution to Eq.(6) is given below:

$$\phi_1(z) = e^{-\frac{1}{2}z(\log(\alpha)+\sqrt{4E_1+(\log\alpha)^2})}C_{11} + e^{-\frac{1}{2}z(\log(\alpha)+\sqrt{4E_1+(\log\alpha)^2})}C_{12}. \tag{7}$$

Using Eq.(7) in Eq.(4), we get the displacement for Case I, i.e.,

$$\tilde{v}_1 = \left(e^{-\frac{1}{2}z(\log(\alpha)+\sqrt{4E_1+(\log\alpha)^2})}C_{11} + e^{-\frac{1}{2}z(\log(\alpha)+\sqrt{4E_1+(\log\alpha)^2})}C_{12}\right)J_1(kr)e^{i\omega t}. \tag{8}$$

Displacement of the upper layer (Case-II)

Again, using the variations from Table 1 in Eq.(5), it reduces to

$$\frac{d^2\phi_1}{dz^2} + \beta\frac{d\phi_1}{dz} - E_2\phi_1 = 0, \tag{9}$$

where: $E_2 = \frac{k^2\tilde{N}_0}{\tilde{L}_0}\left(1 - \frac{c^2}{c_0^2}\right)$; and $c_0^2 = \frac{\tilde{N}_0}{\tilde{\rho}_0}$.

Equation (9) solution can be written as

$$\phi_1(z) = e^{-\frac{1}{2}z(\beta+\sqrt{4E_2+\beta^2})}C_{21} + e^{-\frac{1}{2}z(\beta-\sqrt{4E_2+\beta^2})}C_{22}. \tag{10}$$

Again, using the Eq.(10) in Eq.(4), the displacement for Case II can be derived

$$\tilde{v}_1 = \left(e^{-\frac{1}{2}z(\beta+\sqrt{4E_2+\beta^2})}C_{21} + e^{-\frac{1}{2}z(\beta-\sqrt{4E_2+\beta^2})}C_{22}\right)J_1(kr)e^{i\omega t}. \tag{11}$$

Displacement of the lower half-space

Variations for the half-space are taken from Table 1 and can be used in the Eq.(5), which gives

$$\frac{d^2\phi_2}{dz^2} + \left(\log(\eta) + \frac{\zeta}{1+\zeta z}\right)\frac{d\phi_2}{dz} - E_3\phi_2 = 0, \tag{12}$$

where: $E_3 = \frac{k^2 N_1}{L_1}\left(1 - \frac{c^2}{c_1^2}\right)$; and $c_1^2 = N_1/\rho_1$.

The solution of Eq.(12) is

$$\phi_2(z) = C_3 e^{-\frac{1}{2}z(\log(\eta)+C_1)}\left(H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}+C_1\right] + L_L\left[\frac{\log(\eta)+C_1}{2C_1}, \frac{-C_1}{\zeta}+C_1\right]\right), \tag{13}$$

where: H_U is Hypergeometric U; L_L is Laguerre L; and $C_1 = \sqrt{(4E_3 + \log(\eta)^2)}$.

As $z \to \infty$, so $\phi_2 \to 0$. Hence Eq.(13) reduces to

$$\phi_2(z) = e^{-\frac{1}{2}z(\log(\eta)+C_1+i\omega t)}H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}+C_1\right]C_{33}. \tag{14}$$

Boundary condition

i. $$L_0 \frac{\partial \tilde{v}_1}{\partial z} = 0, \quad \text{at} \quad z = -H \tag{15}$$
(stress free condition)

ii. $$\tilde{v}_1 = \tilde{v}_2 \quad \text{at} \quad z = 0 \tag{16}$$
(Continuity for displacement)

iii. $$L_0 \frac{\partial \tilde{v}_1}{\partial z} = L_1 \frac{\partial \tilde{v}_2}{\partial z} \tag{17}$$
(Stresses are equal at interface)

Dispersion relation for Case I

Using the above boundary conditions, Eqs.(15-17), we have

$$\left(\log(\alpha) + \sqrt{\log(\alpha)^2 + 4E_1}\right) e^{\frac{1}{2}H\left(\log(\alpha)+\sqrt{\log(\alpha)^2+4E_1}\right)} C_{11} + \left(\log(\alpha) - \sqrt{\log(\alpha)^2 + 4E_1}\right) e^{\frac{1}{2}H\left(\log(\alpha)-\sqrt{\log(\alpha)^2+4E_1}\right)} C_{12} = 0 \, , \tag{18}$$

$$C_{11} + C_{12} - H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}\right] C_3 = 0 \, , \tag{19}$$

$$-L_0\left[\log(\alpha) + \sqrt{\log(\alpha)^2 + 4E_1}\right] C_{21} - L_0\left[\log(\alpha) + \sqrt{\log(\alpha)^2 + 4E_1}\right] \times$$
$$\times C_{22} - L_1(\log(\eta) + C_1)\left(H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}\right] + H_U\left[1 + \frac{\log(\eta)+C_1}{2C_1}, 2, \frac{-C_1}{\zeta}\right]\right) C_{33} = 0 \, . \tag{20}$$

Dispersion relation for Case II

Using the above boundary conditions, Eqs.(15-17), we have

$$\left(\beta + \sqrt{4E_2 + \beta^2}\right) e^{\frac{1}{2}H\left(\beta+\sqrt{4E_2+\beta^2}\right)} C_{11} + \left(\beta + \sqrt{4E_2 + \beta^2}\right) e^{\frac{1}{2}H\left(\beta+\sqrt{4E_2+\beta^2}\right)} C_{12} = 0 \, , \tag{21}$$

$$C_{11} + C_{12} - H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}\right] C_{33} = 0 \, , \tag{22}$$

$$-L_0\left(\beta + \sqrt{4E_2 + \beta^2}\right) C_{11} - L_0\left(\beta + \sqrt{4E_2 + \beta^2}\right) C_{12} -$$
$$-L_1(\log(\eta) + C_1)\left(H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}\right] + H_U\left[1 + \frac{\log(\eta)+C_1}{2C_1}, 2, \frac{-C_1}{\zeta}\right]\right) C_{33} = 0 \, . \tag{23}$$

Using Eqs.(15-17), we will get the dispersion relation for both the cases

$$\mathrm{Re}\left(\left|D_{ij}\right|\right) = 0 \, . \tag{24}$$

Table 2. Determinant entities for both Case I and Case II.

D_{ij}	Case I	Case II
D_{11}	$\left(\log(\alpha) + \sqrt{\log(\alpha)^2 + 4E_1}\right) e^{\frac{1}{2}H\left(\log(\alpha)+\sqrt{\log(\alpha)^2+4E_1}\right)}$	$\left(\beta + \sqrt{\beta^2 + 4E_2}\right) e^{\frac{1}{2}H\left(\beta+\sqrt{\beta^2+4E_2}\right)}$
D_{12}	$\left(\log(\alpha) - \sqrt{\log(\alpha)^2 + 4E_1}\right) e^{\frac{1}{2}H\left(\log(\alpha)-\sqrt{\log(\alpha)^2+4E_1}\right)}$	$\left(\beta - \sqrt{\beta^2 + 4E_2}\right) e^{\frac{1}{2}H\left(\beta-\sqrt{\beta^2+4E_2}\right)}$
D_{13}	0	0
D_{21}	1	1
D_{22}	1	1
D_{23}	$-H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}\right]$	$-H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}\right]$
D_{31}	$-L_0\left(\log(\alpha) + \sqrt{\log(\alpha)^2 + 4E_1}\right)$	$-L_0\left(\beta + \sqrt{\beta^2 + 4E_2}\right)$
D_{32}	$-L_0\left(\log(\alpha) - \sqrt{\log(\alpha)^2 + 4E_1}\right)$	$-L_0\left(\beta - \sqrt{\beta^2 + 4E_2}\right)$
D_{33}	$-L_1(\log(\eta) + C_1)\left(H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}\right]\right)$ $-L_1(\log(\eta) + C_1)\left(H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 2, \frac{-C_1}{\zeta}\right]\right)$	$-L_1(\log(\eta) + C_1)\left(H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 1, \frac{-C_1}{\zeta}\right]\right)$ $-L_1(\log(\eta) + C_1)\left(H_U\left[\frac{\log(\eta)+C_1}{2C_1}, 2, \frac{-C_1}{\zeta}\right]\right)$

NUMERICAL AND GRAPH REPRESENTATION

In this part, graphical representations based on frequency Eq.(24) are carried out to examine the effect of heterogeneity parameters in both layers. The subsequent analysis highlights the notable impact of these parameters on the non-dimensional phase velocity c^2/c_0^2 with respect to the non-dimensional wave number kH.

Figure 2 illustrates how the phase velocity (the torsional wave) is affected by inhomogeneity parameters α and β, providing a comparative analysis between Case I and Case II. In both cases, curves 1, 2, and 3 correspond to α values of 2.0, 4.0, and 6.0, in respect. It is evident from the figure that, in both cases, the phase velocity rises with increasing values of the inhomogeneity parameters for a fixed wave number. However, the effect of the parameter β in Case II is more pronounced than that of α in Case I. Notably, the trend in both cases remains consistent: as the values of the inhomogeneity parameters increase, the phase velocity of the torsional wave also increases.

Upper layer: (Gubbins /8/)

N_0 or $\tilde{N}_0 = 6.54\times10^{10}$ N/m^2, L_0 or $\tilde{L}_0 = 7.41\times10^{10}$ N/m^2, ρ_0 or $\tilde{\rho}_0 = 3430$ kg/m^2.

Half-space: (Gubbins /8/)

$N_1 = 7.34\times10^{10}$ N/m^2, $L_1 = 5.98\times10^{10}$ N/m^2, $\rho_1 = 3195$ kg/m^2

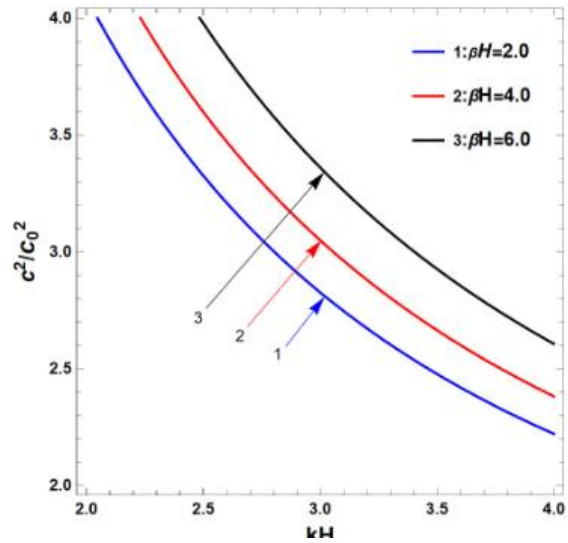

Figure 2. Effect of inhomogeneity parameter β on phase velocity of torsional wave. Case II

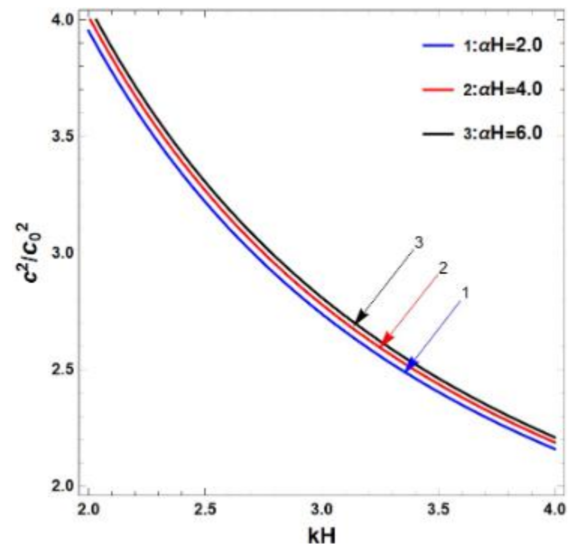

Figure 2. Effect of inhomogeneity parameter α on phase velocity of torsional wave. Case I

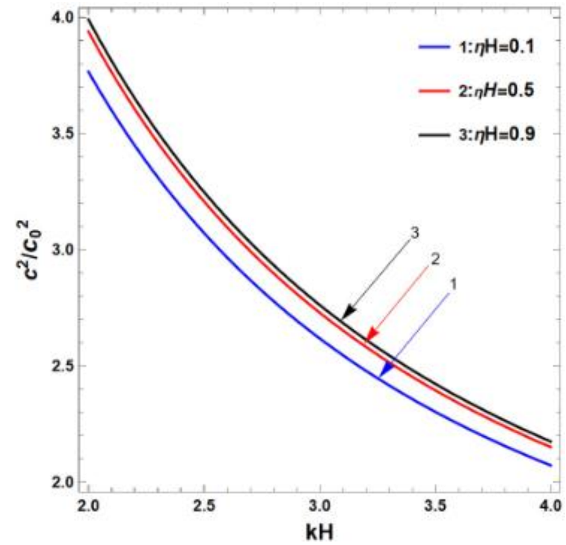

Figure 3. Effect of inhomogeneity parameter η on the phase velocity of torsional wave. Case 1

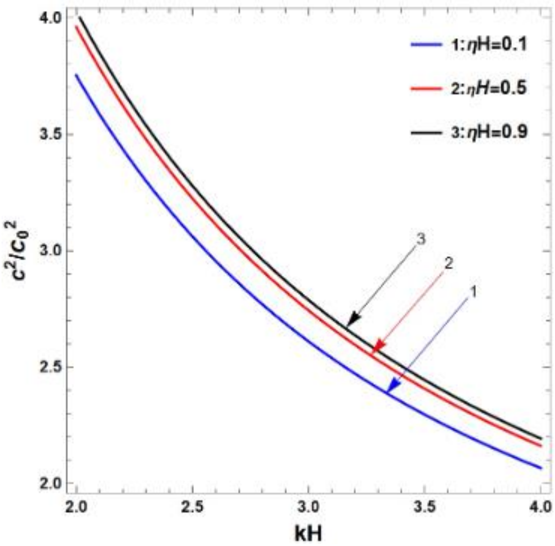

Figure 3. Effect of inhomogeneity parameter η on the phase velocity of torsional wave. Case II

Figure 3 illustrates the inhomogeneity parameter η as examined in Case I and Case II. The graphics show how the dimensionless wave number kH varies in relation to the dimensionless phase velocity c^2/c_0^2. Seeking to investigate how η affects the propagation of torsional waves. Upon analysing the figure, it is observed that the overall trend remains similar in both cases, namely, the phase velocity increases with increasing wave number. The primary distinction between the two cases lies in the fact that the starting and ending points of the curves differ between Case I and Case II.

Figure 4 illustrates the result of phase velocity and the heterogeneity parameter ζ in both Case I and Case II. It has been found from the figure that increasing the values of ζ (0.01, 0.1, 1.0) produces a nearly identical effect in both cases. Similar to the previous figures, the curves exhibit different starting and ending values for each case.

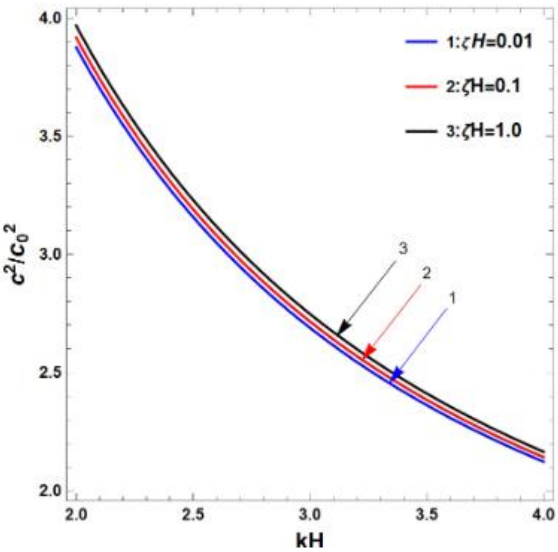

Figure 4. Effect of inhomogeneity parameter ζ on the phase velocity of torsional wave. Case I

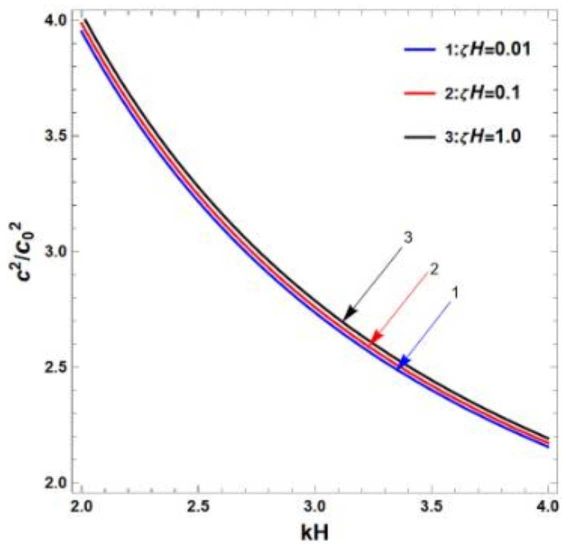

Figure 4. Effect of inhomogeneity parameter ζ on the phase velocity of torsional wave. Case II

CONCLUSION

This study looks into how the inhomogeneity characteristics affect the torsional wave's phase velocity. Using the boundary conditions, the frequency equation is discussed. How the phase velocity is affected by heterogeneity factors α, β, and ζ and η are found to be significant and offer valuable insights into the behaviour of wave propagation in such media.

The following conclusions are made for the current study.

The graphical investigation of Figs. 2 to 4 highlights the significant influence of various inhomogeneity parameters, namely, α, β, η, and ζ of both Case I and Case II.

It is consistently observed that increasing the values of these parameters results in an overall rise in phase velocity, demonstrating a direct relationship between material inhomogeneity and wave propagation speed. While the general trends remain similar across both cases, Case II exhibits a more pronounced sensitivity to inhomogeneity parameters, particularly β, as compared to Case I.

Additionally, despite the similarity in the curve patterns, the differences in the starting and ending values of phase velocity curves for the respective cases reflect variations in the baseline material configurations or boundary conditions. These findings emphasize the importance of inhomogeneity parameters in tailoring wave behaviour and could serve as a basis for designing advanced materials with controlled wave propagation characteristics.

The present investigation can be extended in several directions to obtain more comprehensive understanding of torsional wave propagation in complex geological structures. In future studies, effects of initial stress, temperature variation, porosity and viscoelastic behaviour of the medium may be incorporated to analyse more realistic earth models. Additionally, the influence of irregular interfaces, multi-layered structures and different types of anisotropy can also be considered.

Numerical techniques and advanced analytical methods may be employed to study torsional wave propagation in multi-layered and irregular geological structures. The result obtained in this study may also contribute to the development of improved theoretical models for analysing wave propagation in complex geological media.

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