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ELASTOPLASTIC ANALYSIS IN AN ORTHOTROPIC ROTATING CYLINDER BY USING SETH'S TRANSITION THEORY

ELASTOPLASTIČNA ANALIZA ORTOTROPNOG ROTIRAJUĆEG CILINDRA PRIMENOM TEORIJE PRELAZNIH NAPONA SETA

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Abstract

This article deals with the study of stress distribution in a rotating cylinder made of orthotropic material by using Seth's transition theory. The generalised strain measure is used for finding the governing equation. This model is based on stress-strain relation and on the equilibrium equation. The analytical solution is presented for the rotating cylinder made of orthotropic and isotropic material. The behaviour of stress distribution and angular speed are investigated. From the obtained results, it is noticed that isotropic material cylinder requires higher value of angular speed to yield at the inner surface in comparison to orthotropic material. The hoop stress is maximum at the inner surface for the cylinder made of isotropic/orthotropic material. Results are discussed numerically and graphically.

Izvod

U radu se istražuje raspodela napona kod rotirajućeg cilindra izrađenog od ortotropnog materijala, primenom teorije prelaznih napona Seta. Koristi se generalisana mera deformacije za iznalaženje polazne jednačine. Ovaj model se zasniva na relaciji napon-deformacija i na jednačini ravnoteže. Predstavljeno je analitičko rešenje za rotirajući cilindar od ortotropnog i izotropnog materijala. Istraženo je ponašanje raspodele napona, kao i ugaona brzina. Prema dobijenim rezultatima, uočava se da cilindar od izotropnog materijala zahteva veću ugaonu brzinu za pojavu tečenja na unutrašnjoj površini, u poređenju sa ortotropnim materijalom. Cirkularni napon je maksimalan na unutrašnjoj površini kod cilindra od izotropnog/ortotropnog materijala. Data je numerička i grafička diskusija rezultata.

INTRODUCTION

Structural flexibility is emphasized and applied to an advantage in modern, light-weight designs. Analysis of flexible structures often becomes exceedingly complex due to the difficulties involved in the consideration of finite deformation. In many industrial applications, cylinders are often subjected to extreme operating conditions characterised by high pressure, temperature and corrosive environment. Most cylinders used in industry, are subjected to thermal loads produced by temperature variation in addition to mechanical loads and some applications of also cylinders are subjected to rotation, like centrifugal casting. Research and development reveal that highly orthotropic cylinders can be utilised to achieve a desirable distribution of stress during rotation. Hodge et al. /1/ have investigated the behaviour of rotating cylinder with slowly increasing angular speed which takes it successively through fully elastic, elastoplastic, and fully plastic states. Later, Bhatnagar et al. /2/ have solved the problem for orthotropic cylinders. Zhao et al. /3/ investigated on a thick-walled cylinder under internal pressure. The combined effect of temperature and rotation are calculated for elastoplastic state by Thakur /4/. However, Thakur /5/ have investigated creep deformation in an orthotropic thick-walled cylinder under combined axial load and pressure. Moreover, the elastoplastic analysis in a rotating cylinder with a rigid shaft under mechanical loading are investigated by Prokudin /6/. Temesgen et al. /7/ calculated thermal stress deformation in a functionally graded thick-walled rotating cylinder made of transversely isotropic material and subjected to uniform pressure by using transition theory. The objective of this research paper is to investigate elastoplastic behaviour in an orthotropic rotating cylinder by using transition theory and generalised strain measure.

BASICS EQUATIONS AND FORMULATION OF THE PROBLEM

Let us consider a rotating cylinder with central bore of inner/outer radii a and b and made of orthotropic material and rotating about its axis with angular velocity ω at the inner surface and the density of cylinder is assumed to be constant as show in Fig. 1.

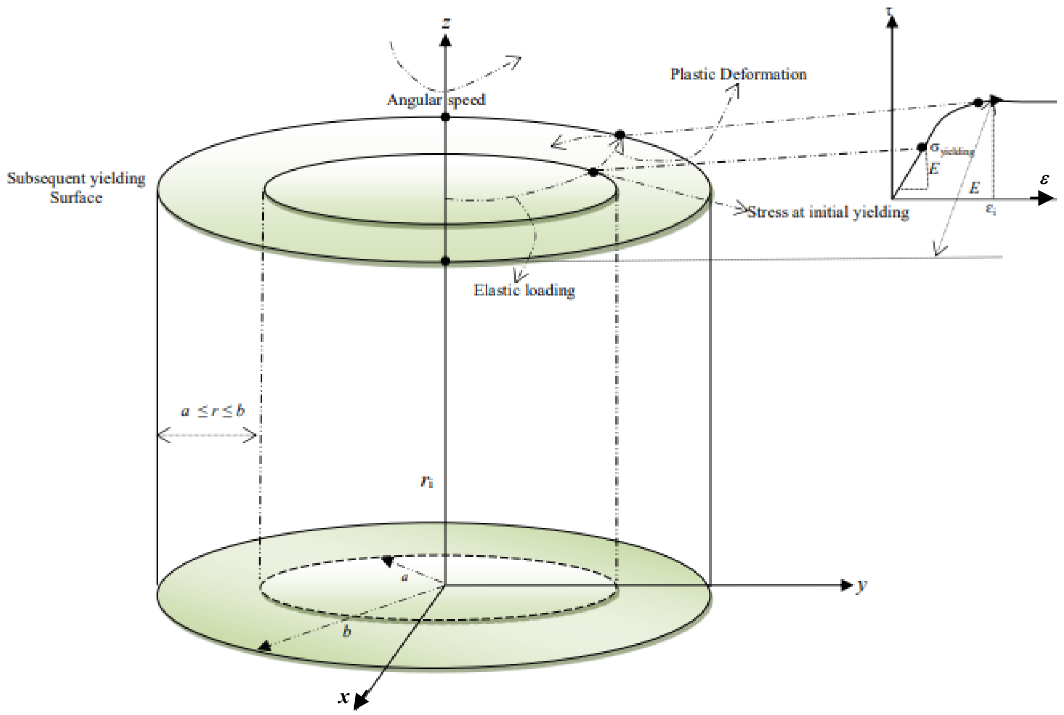

Figure 1. Geometry of the rotating cylinder.

Stress-strain relations for orthotropic material are given by Thakur et al. /4/ and Timoshenko et al. /8/:

$$\tau_{rr}=\frac{c_{11}}{n}[1-(r\eta'+\eta)^n]+\frac{c_{12}}{n}[1-\eta^n]+\frac{c_{13}}{n}[1-(1-d)^n],$$
$$\tau_{\theta\theta}=\frac{c_{21}}{n}[1-(r\eta'+\eta)^n]+\frac{c_{22}}{n}[1-\eta^n]+\frac{c_{23}}{n}[1-(1-d)^n],$$
$$\tau_{zz}=\frac{c_{31}}{n}[1-(r\eta'+\eta)^n]+\frac{c_{32}}{n}[1-\eta^n]+\frac{c_{33}}{n}[1-(1-d)^n],$$
$$\tau_{r\theta}=\tau_{\theta z}=\tau_{zr}=0\,. \tag{1}$$

The equation of equilibrium is given:

$$\frac{d}{dr}(r\tau_{rr})-\tau_{\theta\theta}+\rho\omega^2r^2=0\,. \tag{2}$$

Transition points: inserting Eq.(1) into Eq.(2), we get:

$$\eta T(1+T)^{n-1}\frac{dT}{d\eta}+T(T+1)^n+\frac{c_{12}}{c_{11}}T-\frac{1}{nc_{11}\eta^n}\Big[(c_{11}-c_{21})\times$$
$$\times\{1-\eta^n(T+1)^n\}+(c_{12}-c_{22})(1-\eta^n)+(c_{13}-c_{23})\{1-(1-d)^n\}\Big]-$$
$$-\frac{n\rho\omega^2r^2}{nc_{11}\eta^n}=0\,, \tag{3}$$

where: $r\eta'=\eta T$ (T is function of η, and η is function of r). $T\to\pm\infty$ (elastic to plastic state), and $T\to-1$ (plastic to creep state) /4, 5, 7, 9-28/.

ELASTOPLASTIC DEFORMATION ANALYSIS

Let the transition function ζ be defined as in Thakur et al. /4, 5, 7, 9-28/:

$$\zeta=1-\frac{n}{c_{11}+c_{12}+c_{13}}\left[\tau_{rr}-\frac{n\rho\omega^2r^2}{2}\right]\cong\frac{1}{c_{11}+c_{12}+c_{13}}\times$$
$$\times\left[c_{11}\eta^n(T+1)^n+c_{12}\eta^n+c_{13}(1-d)^n\,\frac{n\rho\omega^2r^2}{2}\right]. \tag{4}$$

By taking logarithmic differentiations and the transition point $T\to\pm\infty$ on Eq.(4), the transition function is given:

$$R=Ar^{-k_1}\,, \tag{5}$$

where: A is constant of integration; and $k_1=(c_{11}-c_{21})/c_{11}$.

From Eqs. (4) and (5), the radial stress is given by:

$$\tau_{rr}=\frac{c_{11}+c_{12}+c_{13}}{n}\left[1-Ar^{-k_1}\right]-\frac{\rho\omega^2r^2}{2}\,. \tag{6}$$

Substituting Eq.(6) into Eq.(2), we get:

$$\tau_{\theta\theta}=\frac{c_{11}+c_{12}+c_{13}}{n}\left[1-A(1-k_1)r^{-k_1}\right]-\frac{\rho\omega^2r^2}{2}\,, \tag{7}$$

and
$$\tau_{zz}=k_2(\tau_{rr}+\tau_{\theta\theta})+[c_{31}-k_2(c_{11}-c_{21})]\frac{k_2}{r}+2k_3, \tag{8}$$

where: $k_2=c_{32}/(c_{12}+c_{22})$; and $k_3=e_{zz}[c_{33}-k_2(c_{11}+c_{21})]$.

By applying boundary condition $\tau_{rr}=0$ at $r=a$, and $r=b$ into Eq.(7), and $\int_a^b r\tau_{zz}dr=0$ into Eq.(8), we get:

$$A=\frac{a^2-b^2}{(a^2-b^2)a^{-k_1}+a^2(b^{-k_1}-a^{-k_1})}\,,$$
$$\frac{c_{11}+c_{12}+c_{13}}{n}=\frac{\rho\omega^2}{2}\left[\frac{(a^2-b^2)a^{-k_1}+a^2(b^{-k_1}-a^{-k_1})}{b^{-k_1}-a^{-k_1}}\right],$$
$$k_3=\frac{1}{2}[k_2(c_{11}+c_{12})-c_{31}]-\frac{\rho\omega^2}{8}(a^2-b^2)k_2\,. \tag{9}$$

Substituting Eq.(9) into Eqs. (6)-(8), we get:

$$\sigma_\theta=\frac{\rho\omega^2b^2}{2Y}\left[\frac{(R_0^2-1)(1+k_1)(R_0^{-k_1}-R^{-k_1})+(1-R_0^{-k_1})(R_0^2-R^2)}{1-R_0^{-k_1}}\right],$$
$$\sigma_r=\frac{\rho\omega^2b^2}{2Y}\left[\frac{(R_0^2-1)(R_0^{-k_1}-R^{-k_1})+(1-R_0^{-k_1})(R_0^2-R^2)}{1-R_0^{-k_1}}\right],$$

and
$$\sigma_z=k_4(\sigma_r+\sigma_\theta)-k_4\frac{\rho\omega^2b^2(1+R_0^2)}{4Y}\,, \tag{10}$$

where: $R_0=a/b$; $R=r/b$; $\sigma_\theta=\tau_{\theta\theta}/Y$; $\sigma_r=\tau_{rr}/Y$; $\sigma_z=\tau_{zz}/Y$; and $k_4=c_{32}/(c_{12}+c_{22})$.

From Eq.(10), we get:

$$\tau_{\theta\theta}-\tau_{rr}=\frac{\rho\omega^2b^2}{2}k_1R^{-k_1}\left[\frac{R_0^2-1}{1-R_0^{-k_1}}\right]. \tag{11}$$

Initial yielding surface: from Eq.(11) it is seen that $|\tau_{\theta\theta}-\tau_{rr}|$ is maximum at $R=R_0$, therefore yielding of the surface becomes:

$$\left|\tau_{\theta\theta}-\tau_{rr}\right|_{R=R_0}=\frac{\rho\omega^2b^2}{2}k_1R_0^{-k_1}\left[\frac{R_0^2-1}{1-R_0^{-k_1}}\right]=Y\,,$$

and angular speed for initial yielding surface is given:

$$\Omega_i^2=\frac{\rho\omega^2b^2}{Y}=\left|\frac{2(1-R_0^{-k_1})}{k_1R_0^{-k_1}(R_0^2-1)}\right|, \tag{12}$$

where: $\omega_i=(\Omega_i/r_0)\sqrt{(Y/\rho)}$.

Subsequent yielding surface: for fully-plastic surface $c_{11}=c_{13}=-c_{12}$ and $c_{23}=c_{21}=-c_{22}$, Eq.(10) becomes:

$$\sigma_\theta=\frac{\Omega^2_{fully-plastic}}{2}\times$$
$$\times\frac{[(R_0^2-1)(1+k_5)(R_0^{-k_5}-R^{-k_5})+(1-R_0^{-k_5})(R_0^2-R^2)]}{1-R_0^{-k_5}}\,,$$

$$\sigma_r=\frac{\Omega^2_{fully-plastic}}{2}\frac{[(R_0^2-1)(R_0^{-k_5}-R^{-k_5})+(1-R_0^{-k_5})(R_0^2-R^2)]}{1-R_0^{-k_5}},$$

and $$\sigma_z=-k_6(\sigma_r+\sigma_\theta)+k_6\frac{\Omega^2_{fully-plastic}(1+R_0^2)}{4}, \tag{13}$$

where: $k_5=(c_{12}-c_{22})/c_{12}$; and $k_6=c_{22}/(c_{12}+c_{22})$.

Isotropic materials: for isotropic materials elastic constants are: $c_{11}=c_{22}=c_{33}$, $c_{12}=c_{21}=c_{32}=c_{23}=c_{31}=c_{13}=c_{11}-2c_{66}$, $\lambda=c_{12}$, $\mu=(c_{11}-c_{12})/2$, and $c=(c_{11}-c_{12})/c_{11}$, Eq.(13) becomes:

$$\sigma_\theta=\frac{\Omega_i^2}{2}\frac{[(R_0^2-1)(1+c)(R_0^{-c}-R^{-c})+(1-R_0^{-c})(R_0^2-R^2)]}{1-R_0^{-c}},$$

$$\sigma_r=\frac{\Omega_i^2}{2}\frac{[(R_0^2-1)(R_0^{-c}-R^{-c})+(1-R_0^{-c})(R_0^2-R^2)]}{1-R_0^{-c}},$$

and $$\sigma_z=\frac{1-c}{2-c}\left[(\sigma_r+\sigma_\theta)-\frac{\Omega_i^2(1+R_0^2)}{4}\right]. \tag{14}$$

Fully-plastic state: for fully-plastic surface (i.e., $c\to 0$), Eq. (14) becomes:

$$\sigma_\theta=\frac{\Omega^2_{fully-plastic}}{2}\frac{[(R_0^2-1)(\log R/R_0)+(\log R_0)(R_0^2-R^2)]}{\log R_0},$$

$$\sigma_r=\frac{\Omega^2_{fully-plastic}}{2}\frac{[(R_0^2-1)(\log R/R_0)+(\log R_0)(R_0^2-R^2)]}{\log R_0},$$

and $$\sigma_z=\frac{1}{2}\left[(\sigma_r+\sigma_\theta)-\frac{\Omega^2_{fully-plastic}(1+R_0^2)}{4}\right]. \tag{15}$$

Validation of results: the present results Eqs.(14)-(15) are the same by neglecting thermal condition from results given by Thakur et al. in /4/.

NUMERICAL RESULTS AND DISCUSSION

To illustrate the analysis, we take numerical values of elastic constants c_{ij} from the existing literature. As a numerical example, elastic constants c_{ij} (units of 10^{10} N/m^2) are given for transversely isotropic material such as (barite: $c_{11}=0.894$, $c_{12}=0.4624$, $c_{22}=0.7842$, $c_{32}=0.2676$) /11/; isotropic material (saturated clay: $c=0.25$ and Poisson's ratio $\nu=0.42857$); $a=1$; $b=2$; and $R_0=0.5$, respectively.

Figure 2 demonstrates the behaviour of angular speed Ω_i^2 versus radii ratios R_0 for the initial yielding surface. It is observed that the cylinder made of isotropic material requires higher value of angular speed to yield at the inner surface in comparison to the orthotropic material.

Figure 3 is prepared to illustrate the behaviour of stress distribution versus radii ratios $R=r/b$. It has been seen that hoop stress is maximum at the inner surface for the cylinder made of isotropic/orthotropic material. Moreover, the rotating cylinder made of isotropic material requires maximum hoop/axial stress as compared to the orthotropic material, and radial stresses at the inner and outer radius vanish, under the imposed boundary conditions.

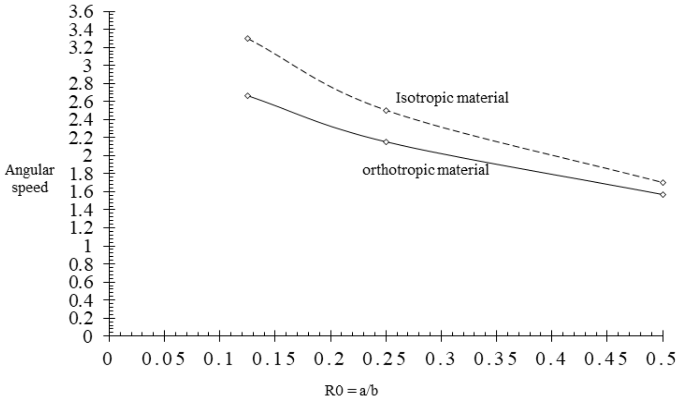

Figure 2. Graph between Ω^2 vs. R_0.

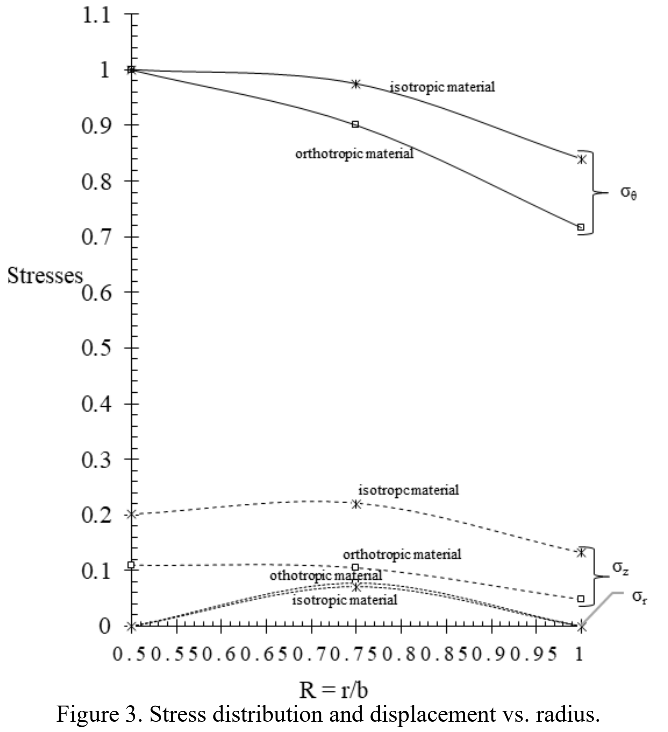

Figure 3. Stress distribution and displacement vs. radius.

CONCLUSIONS

The main findings are given as follows.

- The rotating cylinder made of isotropic material requires higher value of angular speed to yield at the inner surface as compared to the orthotropic material.

- The value of hoop stress is maximum at the inner surface for the cylinder made of isotropic/orthotropic material.
- The cylinder made of isotropic material is more convenient than that of orthotropic material.

Abbreviations

τ_{rr} , $\tau_{\theta\theta}$, τ_{zz} - stress tensor along radial, circumferential and axial direction
a - inner radius
b - outer radius
μ, λ - Lame's constants
k_1, k_2, k_3, k_4, k_5, k_6 - constants (dimensionless)
E - Young modulus
c_{11}, c_{33}, c_{66}, c_{22}, c_{13}, c_{12}, c_{21}, c_{32}, c_{23}, c_{31} - elastic constants
r, θ, z - polar co-ordinates
c - compressibility factor

Greek symbols:
Ω^2 - speed factor (dimensionless)
ρ - density
ω - angular velocity

Non-dimensional quantities
$R_0 = a/b$, $R = r/b$ (radii ratio), $\sigma_r = \tau_{rr}/Y$ (radial stress component), $\sigma_\theta = \tau_{\theta\theta}/Y$ (circumferential stress component), $\sigma_z = \tau_{zz}/Y$ (axial stress component), $\Omega^2 = \rho\omega^2 b^2/Y$ (angular speed).

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