Rekha Panchal[*], *Richa Sharma*

THERMAL ELASTIC-PLASTIC TRANSITION OF FUNCTIONALLY GRADED THICK-WALLED PRESSURIZED ROTATING CYLINDER FABRICATED FROM TRANSVERSELY ISOTROPIC MATERIAL

TERMIČKI ELASTOPLASTIČNI PRELAZNI NAPONI KOD DEBELOZIDOG ROTIRAJUĆEG CILINDRA POD PRITISKOM OD FUNKCIONALNOG KOMPOZITNOG TRANSVERZALNO IZOTROPNOG MATERIJALA

Originalni naučni rad / Original scientific paper
Rad primljen / Paper received: 15.07.2024
https://doi.org/10.69644/ivk-2025-siA-0075

Adresa autora / Author's address:
Jaypee Institute of Information Technology, Noida, India
[*]email: rekhapanchalmath@gmail.com

Keywords
- functionally graded
- elastic
- plastic
- transversely isotropic
- cylinder
- inner pressure
- outer pressure

Abstract

In this paper, elastic-plastic stress analysis is done for a pressurized thick-walled functionally graded cylinder rotating along its axis with angular velocity ω so that the collapse of the cylinder under loading can be avoided. The problem is formulated for functionally graded transversely isotropic material by using Seth's transition theory. Results are analysed theoretically and discussed numerically. From this analysis, it has been concluded that the cylinder made up of functionally graded transversely isotropic material (beryl) has less circumferential stresses than cylinder of transversely isotropic material (magnesium) and isotropic material (steel). Thus, transversely isotropic material (beryl) is safe as compared to the cylinder made up of transversely isotropic material (magnesium) and isotropic material (steel).

Ključne reči
- funkcionalni kompozitni materijal
- elastičnost
- plastičnost
- transverzalno izotropan
- cilindar
- unutrašnji pritisak
- spoljašnji pritisak

Izvod

Izvedena je elastoplastična analiza napona kod debelozidog cilindra po pritiskom koji rotira oko sopstvene ose ugaonom brzinom ω, napravljen od funkcionalnog kompozitnog materijala, tako da se kolaps cilindra pod opterećenjem može izbeći. Problem se formuliše za funkcionalni kompozitni transverzalno izotropni materijal primenom teorije prelaznih napona Seta. Rezultati se analiziraju teorijski uz diskusiju numeričkih podataka. Zaključuje se da se u cilindru od funkcionalnog kompozitnog transverzalnog izotropnog materijala (berilijum) javljaju manji obimski naponi u odnosu na cilindar od transverzalnog izotropnog (magnezijum) i izotropnog materijala (čelik). Stoga je transverzalni izotropni materijal (berilijum) sigurniji u odnosu na transverzalni izotropni (magnezijum) i izotropni materijal (čelik).

INTRODUCTION

Functionally graded materials (FGMs) are non-homogeneous materials. Their physical properties vary throughout the material. Advantages of FGMs over laminated composites include eliminating interfaces between different layers by avoiding points of high-stress concentration. The hollow cylindrical structure can be designed by suitably varying its thermal, mechanical, and physical properties as a function of position within material. The problem of rotating cylinders or disks has wide applications in rotating machinery such as high-speed cameras, gas and steam turbines, planetary landings, etc. The stress analysis of rotating functionally graded cylinders is a popular subject for most of investigators. The solution to the problem of homogeneous isotropic cylinders is discussed in various books /1-2/. Nadai /3/ first studied the deformation behaviour of rotating perfectly plastic cylinders beyond the elastic limits but was unable to satisfy the compatibility requirements. Hodge et al. /4/ solved the problem described by Nadai with a more general condition. Lenard et al. /5/ calculated the instability speed for rotating solid cylinder using the Nadai approach of plastic flow. However, Gamer et al. /6/ gave a new concept of plastic flow and stated that in an edge regime of Tresca's hexagon in principal stress space when yielding occurs in the shaft at the centre of the rotating solid, another plastic region in a side regime rises simultaneously and these two regions spread together into the elastic region with increase in rotation speed. Mack /7/ obtained the analytical solution for the rotating elastic-perfectly plastic solid shaft with axially unconstrained ends in 1991. Lance and Gamer /8/ obtained stresses and radial displacement in rotating linearly hardening hollow shafts with fixed ends in 1983. Mack /7/ treated the problem of rotating elastoplastic hollow shafts with free ends in 1991. In /7/, he investigated the unloading and the secondary flow in a rotating elastic-plastic hollow cylinder. Sharma et al. /10/ investigated thermal elastic-plastic stresses for a rotating functionally graded stainless steel composite cylinder under internal and external pressure with general nonlinear strain-hardening law and von Mises' yield criterion using finite difference method. Analytical solutions obtained by these authors considered yield criterion, linear strain measure, and jump conditions, using the concept of infinitesimal strain theory. The drawback of the classical theory is that it cannot provide the solution for the physical problem of finite deformation. Seth /11/ introduced a new concept of the transition state in his theory called transition theory and subsequently identified the transition state analytically with the asymptotic representation at the turning point of the nonlinear differential equation through a series of papers. The transition theory /11/ is an analytical and most powerful tool of nonlinear analysis which provides a constructive approach for solving various physical problems related to the diverse disciplines of science and engineering. It has been extensively studied, and its concept of generalised strain measure is utilized and extended in the literature by several authors on various problems (see for instance /12-16/). Sharma et al. /14/ analysed thermal hoop stresses in the transversely isotropic thick-walled rotating cylinder under internal pressure. In /14/, he obtained the solution for the elastic-plastic transition of a pressurized transversely isotropic cylinder with steady-state temperature. Aggarwal et al. /15/ investigated safety factors in the transversely isotropic thick-walled cylinder under internal and external pressure using the Lebesgue strain measure and concluded that transversely isotropic material is a better choice for the design of cylinder as compared to isotropic material. Sharma et al. /16/ investigated the thermal creep stresses in rotating pressurized spherical shells and concluded that the effect of non-homogeneity is very pronounced. The present study is directed toward developing an analytical solution for functionally graded transversely isotropic cylinder. Our results extend some of the well-known previous results.

MATHEMATICAL FORMULATION OF PROBLEM

We consider a thick-walled circular cylinder made up of functionally graded transversely isotropic material with inner and outer radii a and b, respectively, subjected to inner pressure P_i and outer pressure P_o. The cylinder is taken so large that the plane sections remain planar during the expansion, and hence the longitudinal strain is the same for all elements at each stage of the expansion.

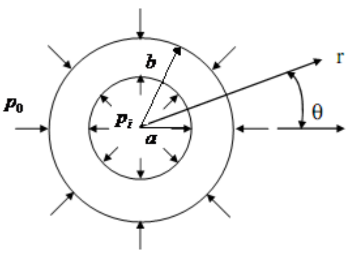

Figure 1. A functionally graded rotating transversely isotropic cylinder with inner pressure P_i and outer pressure P_o at the boundary.

Displacement components in cylindrical polar coordinates are given by

$$u_r = r(1-\beta), \quad u_\theta = 0, \quad u_z = dz\,, \tag{1}$$

where: β is function of $r = \sqrt{(x^2 + y^2)}$ only; and d is a constant.

The finite components of strain /13/ are given as follows:

$$e_{rr} = \frac{1}{n}[1-(r\beta'+\beta)^n], \quad e_{\theta\theta} = \frac{1}{n}[1-\beta^n],$$
$$e_{zz} = \frac{1}{n}[1-(1-d)^n], \quad e_{r\theta} = e_{\theta z} = e_{zr} = 0\,. \tag{2}$$

Stress-strain relations for transversely isotropic material:

$$T_{rr} = C_{11}e_{rr} + (C_{11}-2C_{66})e_{\theta\theta} + C_{13}e_{zz} - \beta_1 T\,,$$
$$T_{\theta\theta} = (C_{11}-2C_{66})e_{rr} + C_{11}e_{\theta\theta} + C_{13}e_{zz} - \beta_2 T\,,$$
$$T_{zz} = C_{13}e_{rr} + C_{13}e_{\theta\theta} + C_{33}e_{zz} - \beta_2 T\,, \; T_{zr} = T_{\theta z} = T_{r\theta} = 0 \tag{3}$$

where: $\beta_1 = C_{11}\alpha_1 + 2C_{12}\alpha_2$; $\beta_2 = C_{12}\alpha_1 + (C_{22} + C_{33})\alpha_2$; α_1 is coefficient of linear thermal expansion along the axis of symmetry; α_2 is coefficient of linear thermal expansion orthogonal to the axis of symmetry; $C_{ij} = C_{0ij}(r/b)^{-k}$; C_{ij} are stiffness constants; and $k \leq 0$ is non-homogeneity parameter.

Using Eqs.(2) in Eqs.(3), we get

$$T_{rr} = \left(C_{011}\left(\frac{r}{b}\right)^{-k}/n\right)[1-(\beta+r\beta')^n] + \left[\left(C_{011}\left(\frac{r}{b}\right)^{-k} - \right.\right.$$
$$\left.\left. -2C_{066}\left(\frac{r}{b}\right)^{-k}\right)/n\right][1-\beta^n] + C_{013}\left(\frac{r}{b}\right)^{-k} e_{zz}\,,$$
$$T_{\theta\theta} = \left[\left(C_{011}\left(\frac{r}{b}\right)^{-k} - 2C_{066}\left(\frac{r}{b}\right)^{-k}\right)/n\right][1-(\beta+r\beta')^n] +$$
$$+\left(C_{011}\left(\frac{r}{b}\right)^{-k}/n\right)[1-\beta^n] + C_{013}\left(\frac{r}{b}\right)^{-k} e_{zz}\,,$$
$$T_{zz} = \left(C_{013}\left(\frac{r}{b}\right)^{-k}/n\right)[1-(\beta+r\beta')^n] + \left(C_{013}\left(\frac{r}{b}\right)^{-k}/n\right)\times$$
$$\times[1-\beta^n] + C_{033}\left(\frac{r}{b}\right)^{-k} e_{zz}\,,$$
$$T_{r\theta} = T_{\theta z} = T_{rz} = 0\,. \tag{4}$$

Equations of equilibrium are all satisfied except,

$$\frac{d}{dr}(T_{rr}) + \left(\frac{T_{rr}-T_{\theta\theta}}{r}\right) + \rho r\omega^2 = 0\,. \tag{5}$$

where: ρ is density of the material. The temperature field satisfying the Laplace equation ($\Delta^2 T = 0$) and $T = T_0$ at $r = a$, $T = 0$ at $r = b$, where T_0 is constant given by

$$T = \frac{T_0 \log\left(\frac{r}{b}\right)}{\log\left(\frac{a}{b}\right)}\,. \tag{6}$$

IDENTIFICATION OF TRANSITION POINT

We know that as the point in the material has yielded, the material at the neighbouring points is on its way to yielding rather than remaining in its complete elastic or fully plastic state. Thus, we can assume that there exists some state in-between elastic and plastic, called as a transition state. So, at transition the differential system defining the elastic state should attain some criticality. The differential equation which comes out to be nonlinear at the transition state is obtained by substituting Eqs.(4) in Eq.(5),

$$nPC_{11}\beta^{n+1}(1+P)^{n-1}\frac{dP}{d\beta} = -nPC_{11}\beta^n(1+P)^n - (C_{11}-2C_{66})nP\beta^n + 2C_{66}\left[\beta^n\{1-(1+P)^n\}\right] - n\beta_1\overline{T_0} + (\beta_2-\beta_1)nr\overline{T_0}\log(\frac{r}{b}) +$$
$$+\rho nr^2\omega^2 - kC_{11}[1-\beta^n(1+P)^n] - k(C_{11}-2C_{66})[1-\beta^n] - kC_{13}[1-(1-d)^n] + n\overline{T_0}\log(\frac{r}{b})k\beta_1\,, \tag{7}$$

where: $r\beta' = \beta P$; and $\overline{T}_0 = T_0/\log(r/b)$.

Critical points or transitional points of Eq.(7) are $P \to -1$ and $P \to \pm\infty$.

The boundary conditions are given by

$$T_{rr} = -p_i \quad \text{at} \quad r = a\,; \quad T_{rr} = -p_o \quad \text{at} \quad r = b. \tag{8}$$

The resultant axial force is given by

$$2\pi\int_a^b rT_{zz}dr = 0\,. \tag{9}$$

MATHEMATICAL APPROACH

The material from the elastic state goes into the plastic state as $P \to \pm\infty$ or to the creep state as $P \to -1$ under inner and outer pressure. It has been shown /7, 8, 10-12/ that the asymptotic solution through the principal stress leads from an elastic to plastic state at the transition point $P \to \pm\infty$. For finding the plastic stress at the transition point $P \to \pm\infty$, we define the transition function TR in terms of T_{rr} as,

$$TR = 2(C_{11}-C_{66}) + nC_{13}e_{zz} - nT_{rr} - n\beta_1 T + B = \\ = \beta^n[C_{11}-2C_{66}+C_{11}(1+P)^n] + B\,. \tag{10}$$

Taking the logarithmic differentiation of Eq.(10) with respect to 'r', with asymptotic value as $P \to \pm\infty$, which on integration yields,

$$TR = Ar^{-C_1}\,. \tag{11}$$

where: A is a constant of integration; and $C_1 = 2C_{066}/C_{011}$.

Using Eq.(11) in Eq.(10), we get

$$T_{rr} = C_3(r) - \beta_0(r)\log\left(\frac{r}{b}\right) - (A/n)r^{-C_1} + (B/n)\,, \tag{12}$$

where: $\beta_0(r) = \beta_1(r)\overline{T}_0$; and

$$C_3(r) = \frac{2\left\{C_{011}\left(\frac{r}{b}\right)^{-k} + C_{012}\left(\frac{r}{b}\right)^{-k}\right\} + nC_{013}\left(\frac{r}{b}\right)^{-k}e_{zz}}{n}\,.$$

Using boundary condition Eq.(8) in the above equation,

$$A = \frac{nb^{C_1}}{\left(\frac{b}{a}\right)^{C_1}-1}\left[p_i - p_0 - C_3(b) + C_3(a) - \beta_0(a)\log\left(\frac{a}{b}\right)\right],$$

and

$$B = n\left[-p_0 - C_3(b) + \frac{1}{\left(\frac{b}{a}\right)^{C_1}-1}\left\{p_i - p_0 - C_3(b) + C_3(a) - \beta_0(a)\log\left(\frac{a}{b}\right)\right\}\right].$$

Substituting the value of A and B in Eq.(12), we get

$$T_{rr} = -p_0 + C_3(r) - C_3(b) - \beta_0(r)\log\left(\frac{r}{b}\right) + \frac{1-\left(\frac{b}{r}\right)^{C_1}}{\left(\frac{b}{a}\right)^{C_1}-1}\left[p_i - p_0 - C_3(b) + C_3(a) - \beta_0(a)\log\left(\frac{a}{b}\right)\right]. \tag{13}$$

Using Eq.(13) in Eq.(5), we have

$$T_{\theta\theta} = (1-k)C_3(r) - (1-k)\beta_0(r)\log\left(\frac{r}{b}\right) - \beta_0(r) - p_0 + \frac{\left(\frac{b}{r}\right)^{C_1}(C_1-1)+1}{\left(\frac{b}{a}\right)^{C_1}-1}\left[p_i - p_0 - C_3(b) + C_3(a) - \beta_0(a)\log\left(\frac{a}{b}\right)\right] + \rho r^2\omega^2\,. \tag{14}$$

The axial stress is obtained from Eq.(4) as

$$T_{zz} = \frac{C_{13}}{2(C_{11}-C_{66})}(T_{rr}+T_{\theta\theta}) + \frac{C_{33}(C_{11}-C_{66}) - C_{13}^2}{(C_{11}-C_{66})}e_{zz} + \frac{T[C_{13}(\beta_1+\beta_2) - 2\beta_2(C_{11}-C_{66})]}{2(C_{11}-C_{66})}\,. \tag{15}$$

From Eqs. (13) and (14), we get

$$|T_{rr} - T_{\theta\theta}| = \left|-kC_3(r) + k\beta_0(r)\log\left(\frac{r}{b}\right) - \beta_0(r) + \frac{C_1\left(\frac{b}{r}\right)^{C_1}}{\left(\frac{b}{a}\right)^{C_1}-1}\left[p_i - p_0 - C_3(b) + C_3(a) - \beta_0(a)\log\left(\frac{a}{b}\right)\right] + \rho r^2\omega^2\right|. \tag{16}$$

From the above equation, it is found that the value of $|T_{rr} - T_{\theta\theta}|$ is maximum at $r = a$, which means yielding of the cylinder will take place at the internal surface. Therefore, we have

$$|T_{rr} - T_{\theta\theta}|_{r=a} = \left|-kC_3(a) + k\beta_0(a)\log\left(\frac{a}{b}\right) - \beta_0(a) + \rho a^2\omega^2 + \frac{C_1\left(\frac{b}{a}\right)^{C_1}}{\left(\frac{b}{a}\right)^{C_1}-1}\left[p_i - p_0 - C_3(b) + C_3(a) - \beta_0(a)\log\left(\frac{a}{b}\right)\right]\right| \equiv Y_i \text{ (say)}\,.$$

The pressure required for initial yielding is given by

$$|P_i| = |P_{ii} - P_{io}| = \left|\frac{p_i - p_0}{Y_i}\right| = \left|\frac{\left(\frac{b}{a}\right)^{C_1}-1}{C_1\left(\frac{b}{a}\right)^{C_1}}\left\{1 - \frac{\rho a^2\omega^2}{Y_i} + \frac{kC_3(a)}{Y_i} + \left(1 - k\log\left(\frac{a}{b}\right)\right)\frac{\beta_0(a)}{Y_i} + \frac{C_1\left(\frac{b}{a}\right)^{C_1}}{\left(\frac{b}{a}\right)^{C_1}-1}\left[\frac{C_3(b) - C_3(a) + \beta_0(a)\log\left(\frac{a}{b}\right)}{Y_i}\right]\right\}\right|, \tag{17}$$

where: $\frac{p_i}{Y_i} - \frac{p_0}{Y_i} = P_{ii} - P_{io} = P_i$.

Using Eq.(17) in Eqs. (12), (13), and (14), we get transitional stresses as

$$\sigma_{rr} = \frac{T_{rr}}{Y_i} = -P_{io} + \frac{C_3(r)}{Y_i} - \frac{C_3(b)}{Y_i} - \frac{\beta_0(r)\log\left(\frac{r}{b}\right)}{Y_i} + \frac{\left(\frac{b}{r}\right)^{C_1}}{\left(\frac{b}{a}\right)^{C_1} - 1}\left[P_i - \frac{C_3(b) + C_3(a) - \beta_0(a)\log\left(\frac{a}{b}\right)}{Y_i}\right], \quad (18)$$

$$\sigma_{\theta\theta} = \frac{T_{\theta\theta}}{Y_i} = (1-k)\frac{C_3(r)}{Y_i} - \frac{(1-k)\beta_0(r)\log\left(\frac{r}{b}\right)}{Y_i} - \frac{\beta_0(r)}{Y_i} - P_{i0} + \frac{\left(\frac{b}{r}\right)^{C_1}(C_1 - 1) + 1}{\left(\frac{b}{a}\right)^{C_1} - 1}\left[P_i - \frac{C_3(b)}{Y_i} + \frac{C_3(a)}{Y_i} - \frac{\beta_0(a)\log\left(\frac{a}{b}\right)}{Y_i}\right] + \frac{\rho r^2\omega^2}{Y_i}, \quad (19)$$

$$\sigma_{zz} = \frac{C_{13}}{2(C_{11} - C_{66})}(\sigma_{rr} + \sigma_{\theta\theta}) + \frac{C_{33}(C_{11} - C_{66}) - C_{13}^2}{C_{11} - C_{66}} e_{zz} + \frac{T[C_{13}(\beta_1 + \beta_2) - 2\beta_2(C_{11} - C_{66})]}{2(C_{11} - C_{66})}. \quad (20)$$

Equations (18)-(20) give elastic-plastic transitional stresses in the thick-walled cylinder under inner and outer pressure.

For a fully plastic state ($C_1 \to 0$), Eq.(16) becomes

$$\left|T_{rr} - T_{\theta\theta}\right|_{r=b} = \left|-kC_3(b) - \beta_o(b) + \beta_o(a) + \frac{1}{\log\left(\frac{b}{a}\right)}\left[p_i - p_0 - C_3(b) + C_3(a)\right] + \rho b^2\omega^2\right| \equiv Y_f \text{ (say)}, \quad (21)$$

where: $P_f = \frac{p_i - p_o}{Y_f} = P_{fi} - P_{fo}$.

The pressure required for fully plastic state is

$$\left|P_f\right| = \left|P_{fi} - P_{f0}\right| = \left|\frac{p_{fi} - p_{fo}}{Y_f}\right| = \left|1 - \frac{\rho a^2\omega^2}{Y_f} + \frac{kC_3(b)}{Y_f} + \frac{\beta_0(b)}{Y_f} + \frac{1}{\log\left(\frac{b}{a}\right)}\frac{C_3(b) - C_3(a)}{Y_f} - \frac{\beta_0(a)}{Y_f}\right|. \quad (22)$$

Now we introduce the following non-dimensional components as:

$R = r/b$, $R_0 = a/b$, $\sigma_{rr} = T_{rr}/Y_i$, $\sigma_{\theta\theta} = T_{\theta\theta}/Y_i$, $\sigma_{zz} = T_{zz}/Y_i$, $\sigma^*_{rr} = T_{rr}/Y_f$, $\sigma^*_{\theta\theta} = T_{\theta\theta}/Y_f$, $\sigma^*_{zz} = T_{zz}/Y_f$, $\Omega^2 = \frac{\rho\omega^2 b^2}{C_{011}}$.

The necessary effective pressure required for initial yielding is given by Eq.(17) in non-dimensional form as

$$\left|P_i\right| = \frac{\left(\frac{1}{R_o}\right)^{C_1} - 1}{C_1\left(\frac{1}{R_o}\right)^{C_1}}\left\{1 - \frac{\rho a^2\omega^2}{Y_i} + \frac{kC_3(a)}{Y_i} + (1 - k\log(R_o))\frac{\beta_0(a)}{Y_i} + \frac{C_1\left(\frac{1}{R_o}\right)^{C_1}}{\left(\frac{1}{R_o}\right)^{C_1} - 1}\left[\frac{C_3(b) - C_3(a) + \beta_0(a)\log(R_o)}{Y_i}\right]\right\}. \quad (23)$$

The transitional stresses given by Eqs. (18) and (19) become

$$\sigma_{rr} = -P_{io} + \frac{C_3(a)}{Y_i} - \frac{C_3(b)}{Y_i} - \frac{\beta_0(r)\log(R)}{Y_i} + \frac{\left(\frac{1}{R}\right)^{C_1}}{\left(\frac{1}{R_o}\right)^{C_1} - 1}\left[P_i - \frac{C_3(b) + C_3(a) - \beta_0(a)\log(R_o)}{Y_i}\right],$$

$$\sigma_{\theta\theta} = (1-k)\frac{C_3(r)}{Y_i} - \frac{(1-k)\beta_0(r)\log(R)}{Y_i} - P_{i0} + \frac{\left(\frac{1}{R_0}\right)^{C_1}(C_1 - 1) + 1}{\left(\frac{1}{R}\right)^{C_1} - 1}\left[P_i - \frac{C_3(b)}{Y_i} + \frac{C_3(a)}{Y_i} - \frac{\beta_0(a)\log(R_0)}{Y_i}\right] + \Omega^2. \quad (24)$$

The effective pressure required for full plasticity is given by

$$\left|P_f\right| = \left|\left\{1 - \frac{\rho a^2\omega^2}{Y_f} + \frac{kC_3(b)}{Y_f} + \frac{\beta_0(b)}{Y_f} + \frac{1}{\log\left(\frac{1}{R_0}\right)}\left[\frac{C_3(b) - C_3(a)}{Y_f}\right] - \frac{\beta_0(a)}{Y_f}\right\}_i\right|. \quad (25)$$

HOMOGENEOUS TRANSVERSELY ISOTROPIC MATERIAL

If we substitute $k = 0$ and $P_o = 0$ in Eqs. (3) and (7), we have

$$C_{ij} = C_{0ij}. \quad (26)$$

Using Eq.(26) in Eq.(24), the radial and circumferential stresses of transversely isotropic cylinder become

$$\sigma_{rr}^{*}=-P_{io}+\frac{C_3(r)}{Y_i}-\frac{C_3(b)}{Y_i}-\frac{\beta_0(r)\log(R)}{Y_i}+\frac{\left(\frac{1}{R}\right)^{C_1}}{\left(\frac{1}{R_o}\right)^{C_1}-1}\left[P_i-\frac{C_3(b)+C_3(a)-\beta_0(a)\log(R_o)}{Y_i}\right], \quad (27)$$

$$\sigma_{\theta\theta}^{*}=-\frac{(1-k)\beta_0(r)\log(R)}{Y_i}-P_{i0}+\frac{\left(\frac{1}{R_0}\right)^{C_1}(C_1-1)+1}{\left(\frac{1}{R}\right)^{C_1}-1}\left[P_i-\frac{C_3(b)}{Y_i}+\frac{C_3(a)}{Y_i}-\frac{\beta_0(a)\log(R_0)}{Y_i}\right]+\Omega^2 . \quad (28)$$

The Eqs. (27) and (28) are the same as those obtained by Sharma et al. /14/ for transversely isotropic cylinders made of homogeneous material without external pressure.

NUMERICAL DISCUSSION

To observe the effect of pressure required for initial yielding and fully plastic state against various radii ratios, Tables 2-4, Figs. 2 and 3, using Table 1 have been drawn.

Table 1. Elastic constants C_{ij} used (in units of 10^{10} N/m^2).

Material	C_{11}	C_{12}	C_{13}	C_{33}	C_{44}	ρ
steel (isotropic)	2.908	1.27	1.27	2.908	0.819	8.05
magnesium (transversely isotropic)	5.97	2.62	2.17	6.17	1.64	1.7364
beryl (transversely isotropic)	2.746	0.98	0.67	4.69	0.883	2.68

To study the impact of pressure needed for initial yielding as well as a full plastic state along with many ratios of radii, as shown in Figs. 2-5 and Tables 2-4 for k = -0.5, -1, -2, respectively, have been drawn.

From Table 2, we can see that the percentage increase in effective pressure required for initial yielding to become fully plastic is very high for cylinder whose radii ratio is 0.2, as compared to radii ratios 0.4 and 0.6 at room temperature. With the introduction of thermal effects, this percentage increases with an increase in temperature.

Also, it has been noticed from Tables 2-4 that this percentage increase in effective pressure required for initial yielding to become fully plastic is high for transversely isotropic material (magnesium) as compared to transversely isotropic (beryl), and isotropic material (steel).

Table 2. Pressure % for initial yielding to reach full plastic state for cylinder made of beryl with ω = 0.5 under P_i = 0.05 and P_o = 0.01.

Temperature	Non-homogeneity k	Pressure needed for initial yielding P_i and full plastic state P_f at various ratio of radii				Required % increase in pressure for initial yielding to reach full plastic state $[(P_f - P_i)/P_f]\times 100$		
		P \ R_0	0.2	0.4	0.6	0.2	0.4	0.6
0	-2	P_i	4.89462	4.42925	3.29211	69.96778726	52.26458305	36.35778412
		P_f	16.2979	9.27875	5.17284			
	-1	P_i	3.0672	2.64306	1.91563	68.45703567	52.25712288	37.93117973
		P_f	9.72388	5.53603	3.0863			
	-0.5	P_i	2.15348	1.74997	1.22739	66.54471336	52.24754207	39.92305546
		P_f	6.43689	3.66467	2.04303			
10	-2	P_i	4.29815	4.10059	3.15175	72.68813583	54.23244341	36.90076498
		P_f	15.7373	8.9596	4.99491			
	-1	P_i	2.72153	2.45396	1.83523	70.2996737	52.96115686	36.89833137
		P_f	9.1633	5.21688	2.90837			
	-0.5	P_i	1.93321	1.63065	1.17697	67.10163351	51.2586982	36.89507265
		P_f	5.87631	3.34552	1.8651			

Table 3. Pressure % for initial yielding to reach full plastic state for cylinder of magnesium with ω = 0.5 under P_i = 0.05 and P_o = 0.01.

Temperature	Non-homogeneity k	Pressure needed for initial yielding P_i and full plastic state P_f at various ratio of radii				Required % increase in pressure for initial yielding to reach full plastic state $[(P_f - P_i)/P_f]\times 100$		
		P \ R_0	0.2	0.4	0.6	0.2	0.4	0.6
0	-2	P_i	5.9604	5.18454	3.76379	70.66732283	55.18476579	41.64169038
		P_f	20.32	11.5687	6.44945			
	-1	P_i	3.89679	3.21184	2.26599	70.72987711	57.62464543	46.37370075
		P_f	13.3132	7.5795	4.22552			
	-0.5	P_i	2.86498	2.22549	1.51709	70.79450709	60.15173029	51.2745901
		P_f	9.80973	5.58491	3.11355			
10	-2	P_i	5.25605	4.80412	3.6035	73.29256457	57.122916	42.31030922
		P_f	19.6801	11.2044	6.24635			
	-1	P_i	3.49575	2.99651	2.17558	72.41620112	58.46936598	45.91351951
		P_f	12.6732	7.21518	4.02241			
	-0.5	P_i	2.61561	2.0927	1.46162	71.47585392	59.91449242	49.78010198
		P_f	9.16981	5.22059	2.91044			

Table 4. Pressure percentage for initial yielding to reach full plastic state for cylinder of steel with $\omega = 0.5$ under $P_i = 0.05$ and $P_o = 0.01$.

Temper-ature	Non-homo-geneity k	Pressure needed for initial yielding P_i and full plastic state P_f at various ratio of radii				Required % increase in pressure for initial yielding to reach full plastic state $[(P_f - P_i)/P_f]\times 100$		
		P \ R_0	0.2	0.4	0.6	0.2	0.4	0.6
0	-2	P_i	5.91168	5.15298	3.74522	70.55818957	54.92337031	41.2331987
		P_f	20.0792	11.4316	6.37302			
	-1	P_i	3.85115	3.18204	2.24817	70.53893819	57.24317337	45.81378029
		P_f	13.072	7.44218	4.14897			
	-0.5	P_i	2.82089	2.19658	1.49964	70.51853245	59.67720914	50.62003201
		P_f	9.56835	5.44749	3.03694			
10	-2	P_i	5.21028	4.77396	3.58547	73.20007818	56.86895243	41.89421109
		P_f	19.4414	11.0685	6.17059			
	-1	P_i	3.45161	2.9674	2.15802	72.24099661	58.08200524	45.31854566
		P_f	12.4342	7.07906	3.94653			
	-0.5	P_i	2.57228	2.06413	1.44429	71.19681453	59.40244317	49.04603987
		P_f	8.93054	5.08437	2.8345			

Table 5. Circumferential stresses for various temperature and non-homogeneity parameters for cylinder made of a) beryl; b) magnesium; c) steel.

	R	Non-homo-geneity k	Transitional circumferential stresses			Full plastic stresses		
			beryl	magnesium	steel	beryl	magnesium	steel
$T_0 = 0$, $P_i = 0.05$, $P_f = 0.01$, $\Omega^2 = 0.5$	0.2	-2	5.50836	5.94074	5.93523	5.48403	5.92334	5.91766
		-1	3.68165	3.99375	3.98813	3.65732	3.97635	3.97055
		-0.5	2.7683	3.02025	3.01458	2.74397	3.00285	2.997
	0.4	-2	7.91873	8.73972	8.71623	7.89656	8.72068	8.69712
		-1	5.33537	5.98625	5.96262	5.3132	5.96721	5.9435
		-0.5	4.0437	4.60952	4.58581	4.02152	4.59048	4.56669
	0.6	-2	9.86238	11.1924	11.139	9.84599	11.177	11.1236
		-1	6.69843	7.82014	7.7665	6.68204	7.80475	7.75109
		-0.5	5.11645	6.13399	6.08026	5.10006	6.1186	6.06485
	0.8	-2	11.6047	13.582	13.4865	11.5939	13.5708	13.4753
		-1	7.95126	9.68799	9.59235	7.94048	9.67678	9.58114
		-0.5	6.12455	7.741	7.64524	6.11377	7.72978	7.63404
$T_0 = 10$, $P_i = 0.05$, $P_f = 0.01$, $\Omega^2 = 0.5$	0.2	-2	4.91217	5.28151	5.27764	4.9535	5.31773	5.31404
		-1	3.33616	3.6207	3.61578	3.3775	3.65691	3.65217
		-0.5	2.54815	2.79029	2.78484	2.58949	2.82651	2.82124
	0.4	-2	7.43605	8.20239	8.18032	7.49852	8.26631	8.24424
		-1	5.05454	5.67934	5.65636	5.11701	5.74326	5.72028
		-0.5	3.86379	4.41782	4.39438	3.92626	4.48173	4.4583
	0.6	-2	9.55063	10.8419	10.7895	9.61085	10.9086	10.856
		-1	6.5245	7.62693	7.57378	6.58472	7.69366	7.64036
		-0.5	5.01143	6.01944	5.96595	5.07165	6.08617	6.03252
	0.8	-2	11.4817	13.439	13.3441	11.5291	13.4968	13.4017
		-1	7.89779	9.62432	9.52895	7.94522	9.6822	9.58655
		-0.5	6.10584	7.717	7.62139	6.15327	7.77488	7.67899

It has been observed that with a decrease in non-homogeneity (k = -0.5 to -2) pressure required for initial yielding increases. It has been observed from Fig. 2 that the pressure required for initial yielding in a functionally graded rotating cylinder under pressure is maximum at the internal surface. It has also been noticed that the effective pressure required for initial yielding to become fully plastic is high for less non-homogeneous material as compared to rotating cylinder made of highly non-homogeneous material. It is noticed in Figs. 2 and 3 that with an increase of nonlinearity, pressure required for initial yielding increases significantly for rotating cylinder of isotropic and transversely isotropic materials. The pressure required for initial yielding for the fully plastic state is maximum at the internal surface as illustrated in Figs. 4 and 5. As the angular speed of the cylinder increases the pressure required for the initial yielding and fully plastic state increases, as can be seen in Figs. 2 to 5. Also, to calculate the transitional circumferential and fully plastic stresses based on the above analysis, definite integrals in Eqs. (23) to (25) have been evaluated by the use of Mathematica®. To discuss the effects of transitional and circumferential stresses in the thick-walled transversely isotropic cylinder under pressure, Table 3 and Figs. 6 to 10 are made for various temperature and non-homogeneity parameter between stresses and radii ratio $R = r/b$.

It is observed from Table 3 that for transitional and fully plastic state, circumferential stress is maximum at the external surface. Also, circumferential stresses are less for the highly non-homogeneous cylinder as compared to the less non-homogeneous cylinder of transversely isotropic and isotropic material. These circumferential stresses decrease with increase in temperature. Fully plastic stresses decrease with the introduction of temperature. It can be seen from Figs. 6 and 7 that transitional circumferential stress increases with increase in nonlinearity. These stresses increase with the increase in angular velocity as can be seen in Figs. 6 to 8. These circumferential stresses increase with the increase in inner pressure as illustrated in Fig. 8. Also, circumferential stresses are less for cylinder made of beryl, as compared to magnesium and steel. It is noticed from Figs. 9 to 10 that fully plastic stresses for the highly homogeneous cylinder are less as compared to a less non-homogeneous cylinder. It can be seen from Fig. 10 that with the increase in nonlinearity, fully plastic stresses increase.

R_0

R_0

R_0

Figure 2. Initial yielding effective pressure needed for cylinders (beryl, magnesium, steel) with $N = 3$.

R_0

R_0

R_0

Figure 3. Initial yielding effective pressure needed for cylinders (beryl, magnesium, steel) with $N = 7$.

R_0

R_0

R_0

Figure 4. Effective pressure needed for full plasticity for cylinders (beryl, magnesium, steel) with $N = 3$.

R_0

R_0

R_0

Figure 5. Effective pressure needed for full plasticity for cylinders (beryl, magnesium, steel) with $N = 7$.

R

R

R

Figure 6. Transitional circumferential stresses for cylinders (beryl, magnesium, steel) with P_i = 0.05, P_o = 0.01, and N = 3.

R

R

R

Figure 7. Transitional circumferential stresses for cylinders (beryl, magnesium, steel) with P_i = 0.05, P_o = 0.01, and N = 7.

R

R

R

Figure 8. Transitional circumferential stresses for cylinders (beryl, magnesium, steel) with P_i = 1, P_o = 0.01, and N = 7.

R

R

R

Figure 9. Full plastic circumferential stresses for cylinders (beryl, magnesium, steel) with P_i = 0.05, P_o = 0.01, and N_g = 3.

R

R

R

Figure 10. Full plastic circumferential stresses for cylinders (beryl, magnesium, steel) with $P_i = 0.05$, $P_o = 0.01$, and $N = 7$.

CONCLUSIONS

The following conclusions are derived from the elastic-plastic stress analysis of a functionally graded transversely isotropic rotating cylinder.

- Plastic yielding occurs first at the inner surface where the stress difference is maximum.
- Circumferential stresses and fully plastic stresses at the outer surface are found to be maximum.
- If the inner pressure of the thick cylinder is increased sufficiently, yielding will occur starting at the inner surface and spread outwards until the whole cross-section becomes plastic.
- Highly functionally graded material has less circumferential stresses as compared to less functionally graded material.
- Introduction of thermal effects decreases the circumferential stresses.

Cylinder made of transversely isotropic material (beryl) is a better choice for design as compared to transversely isotropic (magnesium) and isotropic material (steel), because circumferential stress is less for beryl as compared to magnesium and steel.

REFERENCES

1. Chakrabarty, J., Applied Plasticity, 2nd Ed., Springer, New York, NY, 2010. doi: 10.1007/978-0-387-77674-3
2. Hetnarksi, R.B., Ignaczak J., The Mathematical Theory of Elasticity, 2nd Ed., CRC Press, Boca Raton, 2011. doi: 10.1201/978 1439828892
3. Nadai, A., Theory of Flow and Fracture of Solids, McGraw Hill Book Co., New York, 1950.
4. Hodge Jr., P.G., Balaban, M. (1962), *Elastic-plastic analysis of a rotating cylinder*, Int. J Mech. Sci. 4(6): 465-476. doi: 10.101 6/S0020-7403(62)80008-3
5. Lenard, J., Haddow, J.B. (1972), *Plastic collapse speeds for rotating cylinders*, Int. J Mech. Sci. 14(5): 285-292. doi: 10.1016/ 0020-7403(72)90084-7
6. Gamer, U., Sayir, M. (1984), *Elastic-plastic stress distribution in a rotating solid shaft*, J. App. Math. Phy. 35(6): 601-617. doi: 10.1007/BF00952107
7. Mack, W. (1991), *Rotating elastic-plastic tube with free ends*, Int. J Solids Struct. 27(11): 1461-1476. doi: 10.1016/0020-768 3(91)90042-E
8. Gamer, U., Lance, R.H. (1983), *Stress distribution in a rotating elastic-plastic tube*, Acta Mech. 50(1-2): 1-8. doi: 10.1007/bf0 1170437
9. Lindner, T., Mack, W. (1998), *Residual stresses in an elastic-plastic shaft with fixed ends after previous rotation*, J Appl. Math. Mech. 78(2): 75-86. doi: 10.1002/(SICI)1521-4001(1998 02)78:2%3C75::AID-ZAMM75%3E3.0.CO;2-V
10. Sharma, S., Yadav, S. (2013), *Thermo elastic-plastic analysis of rotating functionally graded stainless steel composite cylinder under internal and external pressure using finite difference method*, Adv. Mater. Sci. Eng. 2013: Art. ID 810508. doi: 10.1 155/2013/810508
11. Seth, B.R. (1970), *Transition conditions: The yield condition*, Int. J Non-linear Mech. 5(2): 279-285. doi: 10.1016/0020-7462 (70)90025-9
12. Borah, B.N. (2005), *Thermo-elastic-plastic transition*, Contemp. Math. 379: 93-111. doi: 10.1090/conm/379/07027
13. Seth, B.R. (1966), *Measure-concept in mechanics*, Int. J Non-linear Mech. 1(1): 35-40. doi: 10.1016/0020-7462(66)90016-3
14. Sharma, S., Sahni, M., Kumar, R. (2009), *Thermo elastic-plastic transition of transversely isotropic thick-walled rotating cylinder under internal pressure*, Adv. Theor. Appl. Mech. 2(3): 113-122.
15. Aggarwal, A.K., Sharma, R., Sharma, S. (2013), *Safety analysis using Lebesgue strain measure of thick-walled cylinder for functionally graded material under internal and external pressure*, The Sci. World J. 2013: Art. ID 676190. doi: 10.1155/2013/67 6190
16. Sharma, S., Panchal, R. (2017), *Thermal creep deformation in pressurized thick-walled functionally graded rotating spherical shell*, Int. J Pure Appl. Math. 114(3): 435-444. doi: 10.12732/ij pam.v114i3.2

Nomenclature of symbols

a, b : inner and outer radii of cylinder
u_r, u_θ, u_z : displacement components
d : constant
R : radial distance
x, y, z : Cartesian coordinates
e_{ii} : first strain invariant
r, θ, z : cylindrical polar coordinates
β : function of r only
e_{ij} and T_{ij} : strain and stress tensor
P : function of β only
C_{ij} : material constants
λ and μ : Lame's constants
$\sigma_{rr} = T_{rr}/C_{11}$: radial stress
$R = r/b$, $R_0 = a/b$
$\sigma_{\theta\theta} = T_{\theta\theta}/C_{11}$: circumferential stress
$\sigma_{zz} = T_{zz}/C_{11}$: axial stress
ω : angular velocity
ρ : density