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ANALYTICAL SOLUTION OF BENDING SHEET MADE OF CAST IRON/BRONZE MATERIAL

ANALITIČKO REŠENJE SAVIJANJA PLOČE IZRAĐENE OD LIVENOG GVOŽĐA/BRONZE

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Abstract

This article deals with the study of stress distribution in a rectangular bending sheet made of cast iron/bronze material by using transition theory. The extension and contraction regions are presented by transition points of the differential equation in defining the deformed fields. Mathematical modelling is based on the stress-strain relation and equilibrium equation. Analytical solutions are presented for the bending sheet made of cast iron/bronze material. It has been observed that circumferential stress is maximal at the inner surface for the extension region, whereas reverse results are obtained in case of the compression region.

Izvod

U ovom radu proučava se raspodela napona kod pravougaone savijene ploče, izrađene od livenog gvožđa/bronze, primenom teorije prelaznih napona. Oblasti izduženja i skraćenja su predstavljene prelaznim tačkama u diferencijalnoj jednačini pri definisanju deformacionih polja. Matematičko modeliranje se zasniva na relaciji napon-deformacija i na jednačini ravnoteže. Analitička rešenja su predstavljena kod savijene ploče izrađene od livenog gvožđa/bronze. Uočava se da je cirkularni napon maksimalan na unutrašnjoj površini u oblasti izduženja, dok je obrnuta situacija u slučaju oblasti skraćenja.

INTRODUCTION

Creep is the time-dependent deformation below the yield strength of the material under constant stress and is known to be quite dominant under high temperatures, especially with metals. It is a high-temperature slow deformation due to repeated stress and their rates are necessary for evaluating boiler materials, jet engines, sheets, gas turbines and high-temperature application. The behaviour of metals helps to design systems that are defiant to failure. Shigeru et al. /1/ examined perforated sheets with randomly distributed holes set up as the plane models of damaged materials under the condition of biaxial tension. Liegard et al. /2/ have investigated stress distribution in a sheet metal by using large-scale four-point bending test.

The objective of this research paper is to investigate the stress distribution in a bending sheet made of cast iron/ bronze material with extension/compression region by using the concept of generalised strain measures and asymptotic solution through the principal stresses-difference. Results are presented graphically and discussed.

GOVERNING EQUATION

We consider a rectangular sheet bent into the form of circular cylinder. The bending moment of couple M per unit length is perpendicular to the plane of the paper. Let the unstrained planes (x', y') after the deformation of the sheet remain orthogonal to each other. Displacement components are given by, /3/:

$$u = x - x' = x - f_1(\xi), \quad v = y - y' = y - f_2(\eta) \tag{1}$$

where: $z = x - iy = F(\xi + i\eta) = F(\zeta)$; and f_1, f_2, F are functions of ξ, η, respectively, which have to be determined.

The generalised strain components are, /3 -5/,

$$\varepsilon_{\xi\xi} = \frac{1}{n^m}\left[1 - F_2^{n/2} f_1'^{n}\right]^m, \quad \varepsilon_{\eta\eta} = \frac{1}{n^m}\left[1 - F_2^{n/2} f_2'^{n}\right]^m,$$
$$\varepsilon_{\xi\eta} = 0, \tag{2}$$

where: $F_2 = (d\xi/dz)^2$.

The stress-strain relations for isotropic material are given by, /6/:

$$T_{ij} = \lambda\delta_{ij} I_1 + 2\mu\varepsilon_{ij}, \quad (i, j = 1,2,3). \tag{3}$$

Substituting Eq.(2) into Eq.(3), we get:

$$\tau_{\xi\xi} = \lambda\varepsilon_{ii} + \frac{2\mu}{n^m}\left[1 - F_2^{n/2} f_1'^{n}\right]^m,$$
$$\tau_{\eta\eta} = \lambda\varepsilon_{ii} + \frac{2\mu}{n^m}\left[1 - F_2^{n/2} f_2'^{n}\right]^m, \quad \tau_{\xi\eta} = 0, \tag{4}$$

where: $\varepsilon_{ii} = \frac{1}{n^m}\left[1 - F_2^{n/2} f_1'^{n} + \left[1 - F_2^{n/2} f_2'^{n}\right]^m\right]^m$.

The equations of equilibrium in ξ and η coordinates are given:

$$\frac{\partial\tau_{\xi\xi}}{\partial\xi} - \frac{1}{2}(\tau_{\xi\xi} - \tau_{\eta\eta}) - \frac{\partial}{\partial\xi}(\log F_2) = 0,$$
$$\frac{\partial\tau_{\eta\eta}}{\partial\eta} - \frac{1}{2}(\tau_{\eta\eta} - \tau_{\xi\xi}) - \frac{\partial}{\partial\eta}(\log F_2) = 0, \tag{5}$$

where: $F_2 = h^2 = |d\zeta/dz|$; $A = 1 - F_2^{n/2} f_1'^{n}$ and $B = 1 - F_2^{n/2} f_2'^{n}$.

Equation (5) can be written as

$$\frac{\partial}{\partial\xi}\left[\lambda(A^m+B^m)+2\mu A^m\right]-\mu(A^m-B^m)\frac{\partial}{\partial\xi}(\log F_2)=0\,,$$

$$\frac{\partial}{\partial\eta}\left[\lambda(A^m+B^m)+2\mu B^m\right]-\mu(B^m-A^m)\frac{\partial}{\partial\eta}(\log F_2)=0\,. \quad (6)$$

Investigation of transition points: Eq.(6) reduced as

$$\frac{\partial}{\partial\xi}\left[\ln\frac{(A^m-B^m)}{F_2^{c/2}}\right]+\frac{2-c}{(A^m-B^m)}\frac{\partial B^m}{\partial\xi}=0\,, \quad (7)$$

$$\frac{\partial}{\partial\eta}\left[\ln\frac{(A^m-B^m)}{F_2^{c/2}}\right]-\frac{2-c}{(A^m-B^m)}\frac{\partial A^m}{\partial\eta}=0\,. \quad (8)$$

The transition points, /7-16/, from Eqs. (7) and (8) are $f'_1 \to 0$ in the extension region, and $f'_1 \to 0$ in compression region.

PROBLEM SOLUTION

Extension region: integrating Eqs.(7)-(8) with respect to η and ξ, we get:

$$A^m-B^m \cong K_1(\xi)F_2^{c/2}\,, \quad (9)$$

and

$$A^m-B^m \cong K_2(\eta)F_2^{c/2}(1-B^m)^{2-c}\,, \quad (10)$$

where: $K_1(\xi)$ and $K_2(\eta)$ are function of ξ and η only. From Eqs. (9) and (10), we obtain as:

$$1-B^m \cong \left[\frac{K_1(\xi)}{K_2(\eta)}\right]^{1/(2-c)}\,. \quad (11)$$

Since $B^m = (1-\phi'^n F_2^{n/2})^m$ and ϕ' is a function of η only, F_2 should be the product of a function of ξ and a function of η. Hence,

$$F_2=\left|\frac{d\xi}{dz}\right|^2=F'(\xi)\bar{F}'(\bar{\xi})=L_1(\xi)L_2(\eta)\,, \quad (12)$$

$$\left(\frac{\partial^2}{\partial\xi^2}-\frac{\partial^2}{\partial\bar{\xi}^2}\right)\left[\log\frac{F'(\xi)}{\bar{F}'(\bar{\xi})}\right]=0\,.$$

Thus, we get

$$F(y)=K_0\exp(k\xi)\,, \quad (13)$$

which shows that the deformed sheet can take a circular form, and hence

$$z=F'(\xi)=\exp(\xi)\,. \quad (14)$$

Now, ϕ' is to be a constant, so that

$$\varphi'=A_0\eta+B_0\,,$$

where: A_0 and B_0 being constants. Also, $\xi = \log z$, $r = \exp(\xi)$, $\eta = \theta$, $F_2 = \exp(-2\xi) = r^{-2}$.

From Eqs. (9) and (10) we see that

$$A^m-B^m=D_0r^{-c}\left[1-(1-A_0^n r^{-n})^m\right]^{(2-c)}\,, \quad (15)$$

where: D_0 being a constant. Again, the second stress invariant is proportional to $(\tau_{rr}-\tau_{\theta\theta})^2$. Now,

$$\tau_{\xi\xi}-\tau_{\eta\eta}=\tau_{rr}-\tau_{\theta\theta}=2\mu n^{-m}(A^m-B^m)=$$

$$=D_1r^{-c}\left[1-(1-A_0^n r^{-n})^m\right]^{(2-c)}\,, \quad (16)$$

where: D_1 is a constant. Here, we take only one stage of creep which corresponds to $m = 1$. Thus, we have

$$\tau_{rr}-\tau_{\theta\theta}=D_2r^{-2n+c(n-1)}\,. \quad (17)$$

The equation of equilibrium gives

$$\tau_{rr}=\frac{D_2}{2n-c(n-1)}r^{-2n+c(n-1)}+D_3\,.$$

If the outer boundary of the strip $r = b$ is free from pressure, we get

$$\tau_{rr}=\frac{D_2}{2n-c(n-1)}\left[\left(\frac{b}{r}\right)^{2n-c(n-1)}-1\right]. \quad (18)$$

Compression region: integrating Eqs.(7)-(8) with respect to η and ξ, we get

$$A^m-B^m \cong L_1(\eta)F_2^{c/2} \cong L_2(\xi)F_2^{c/2}(A^m-1)^{2-c}\,. \quad (19)$$

Here, we see that F_2 is again to be a product of a function of ξ and a function of η. Since $A^m = (1-f'^n F_2^{n/2})^m$ and f' is a function of ξ only, the factor $(A^m-1)^{2-c}$ can be absorbed in $L_2(\xi)$. $L_1(\xi)$ become a constant H_0 and we get

$$A^m-B^m=H_0r^{-c}\,. \quad (20)$$

The second stress invariant now becomes

$$\tau_{\xi\xi}-\tau_{\eta\eta}=\tau_{rr}-\tau_{\theta\theta}=H_1r^{-c}\,, \quad (21)$$

where: H_1 is a new constant. Also, $c \to 0$, we get

$$\tau_{rr}-\tau_{\theta\theta}=H_1=Y\,. \quad (22)$$

Assuming the inner surface of the sheet $r = a$ to be free from traction and using the equilibrium Eq.(5), we get

$$\tau_{rr}=\frac{Y}{c}\left[\left(\frac{a}{r}\right)^c-1\right]. \quad (23)$$

From Eq.(17) we note that when $c \to 0$, $n \to 0$,

$$\tau_{rr}-\tau_{\theta\theta}=D_2=-Y\,.$$

Hence,

$$\tau_{rr}=\frac{Y}{2n-c(n-1)}\left[1-\left(\frac{b}{r}\right)^{2n-c(n-1)}\right]. \quad (24)$$

For the neutral boundary, $r = d$, we equate Eq.(23) and Eq.(24):

$$1+\frac{c}{2n-c(n-1)}=\left(\frac{a}{d_0}\right)^c+\frac{c}{2n-c(n-1)}\left(\frac{b}{d_0}\right)^{2n-c(n-1)}\,. \quad (25)$$

For $n \to 0$, this is the stationary stage

$$d_0^c=\frac{1}{2}(a^c+b^c)\,. \quad (26)$$

For $c \to 0$ this is always satisfied.

Couple applied to the end: for the contraction region

$$\tau_{\theta\theta}=\frac{Y'}{c}\left[\left(\frac{a}{r}\right)^c-1\right]-Y\,. \quad (27)$$

For the extension region

$$\tau_{\theta\theta}=\frac{Y}{2n-c(n-1)}\left[1-\left(\frac{b}{r}\right)^{2n-c(n-1)}\right]+Y\,. \quad (28)$$

The resultant force over a section is

$$\int_a^b\tau_{\theta\theta}dr=\int_a^b\frac{d}{dr}(r\tau_{rr})dr=\left[r\tau_{rr}\right]_a^b=0\,. \quad (29)$$

The moment of couple per unit widths on the end is given:

$$M=\int_a^b\tau_{\theta\theta}rdr=\int_a^b\frac{d}{dr}(r\tau_{rr})rdr=\left[r^2\tau_{rr}\right]_a^b-\int_a^b r\tau_{rr}dr=-\int_a^b r\tau_{rr}dr=$$

$$=\frac{Y}{C}\left[\frac{d_0^2-a^2}{2}-\frac{a^C(d_0^{2-C}-a^{2-C})}{2-C}-\frac{C}{2n-C(n-1)}\left\{\frac{b^2-d_0^2}{2}-\right.\right.$$

$$\left.\left.-\frac{b^{2n-C(n-1)}b^{(1-n)(2-C)}-d^{(1-n)(2-C)}}{(2-C)(1-n)}\right\}\right]. \quad (30)$$

NUMERICAL ILLUSTRATION AND DISCUSSION

To investigate the effect of stress distribution and radii ratio $R = r/b$ for the bending sheet made of isotropic materials (*say:* cast iron $\nu = 0.27$; bronze $\nu = 0.34$) by /6/, the following numerical values are taken: $a = 1$ (inner radius), $b = 2$ (outer radius), and $r \in (a, b)$.

In Fig. 1, the curves are drawn between stress distributions versus radii ratio required for the initial yielding stage/fully-plastic state in the extension/compression region. It is observed that the circumferential stress is maximum at the inner surface for extension region, whereas reverse results are obtained in case of compression region. Furthermore, in the neutral axis, sheet made of cast iron requires maximum circumferential stress in comparison to the sheet made of bronze material for initial/fully-plastic state. Moreover in the compression region, stresses must be reduced at the neutral axis, but reverse results are obtained in the extension region of the sheet made of cast iron/bronze material.

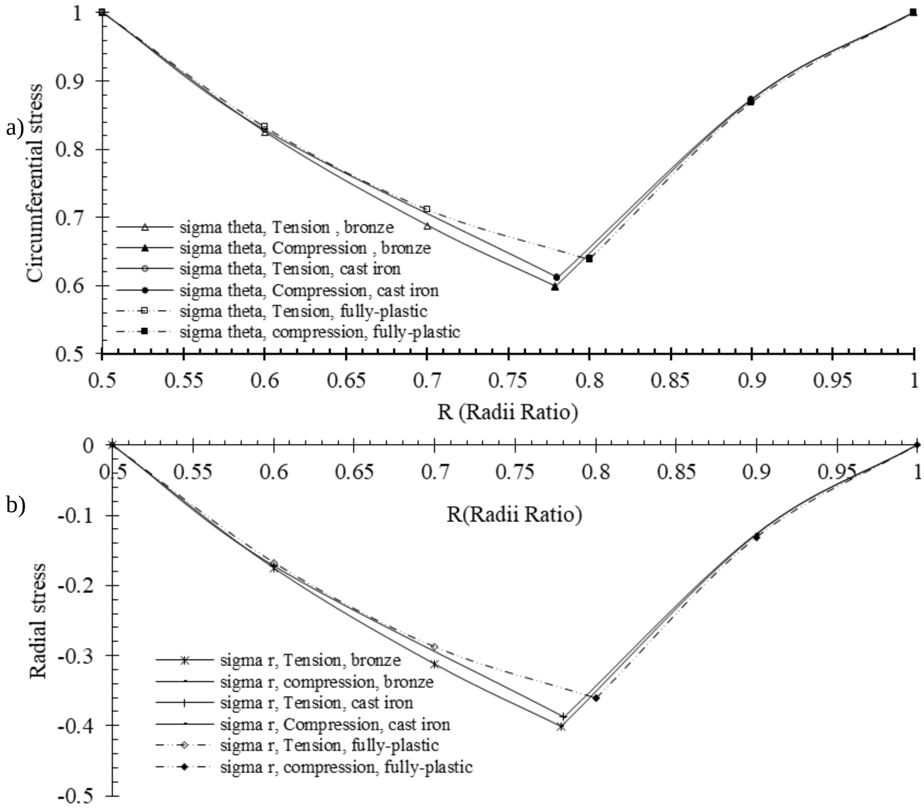

Figure 1. Graphical comparisons between stress distributions vs. radii ratio: a) circumferential stress; and b) radial stress.

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