Gaurav Verma[1*], *Bachittar Singh*[2]

MATHEMATICAL MODEL OF SPHERICAL SHELL UNDER CREEP DEFORMATION
MATEMATIČKI MODEL DEFORMACIJE PUZANJA SFERNE LJUSKE

Originalni naučni rad / Original scientific paper
UDK /UDC:

Rad primljen / Paper received: 24.11.2022

Adresa autora / Author's address:
[1)] Department of Mathematics, Hans Raj Mahila Maha Vidyalaya, Jalandhar, Punjab, India
[*]email: gkdon85@gmail.com
[2)] Khalsa Institute of Management and Technology for Women, Ludhiana, Punjab, India bachittar.singh88@gmail.com

Keywords
- creep stress
- strain rates
- spherical shell
- pressure, compressibility
- measure

Abstract

Creep response of spherical shells under pressure is investigated to provide design guidelines to make the mathematical model applicable in engineering fields. In this study, creep deformation of the spherical shell is impacted by internal pressure and compressibility. Seth's transition theory is adopted to derive the governing equations of creep deformation of the pressurized spherical shell without using the ad hoc assumptions of classical theory of deformation. Pressure required to initiate creep behaviour in the spherical shell made up of different thickness ratio is studied. Results are investigated theoretically and presented graphically.

Ključne reči
- napon puzanja
- brzina deformacije
- sferna ljuska
- pritisak, stišljivost
- mera

Izvod

Istraženo je puzanje u sfernim ljuskama pod pritiskom, radi dobijanja preporuka u projektovanju, kako bi matematički model imao inženjersku primenu. U ovom radu deformacije puzanja u sfernoj ljuski izazivaju unutrašnji pritisak i stišljivost. Primenjuje se Setova teorija prelaznih napona za izvođenje jednačina deformacija puzanja sferne ljuske pod pritiskom, ali izbegavanjem ad hoc pretpostavki koje nudi klasična teorija deformacija. Proučen je pritisak potreban za iniciranje procesa puzanja u sfernoj ljuski, izvedenoj prema različitim odnosima debljina. Rezultati su analizirani teorijski i predstavljeni su grafički.

INTRODUCTION

In practical engineering, the spherical shells are exposed to complex and diverse boundary conditions, and hence, their creep analysis is very important and critical in their structural design. Much research has been already conducted on the cylindrical, conical, and spherical shells on the dependence of creep deformation. Researchers reviewed several nonlinear problems of shell structures, their creep deformation and stability. New scientific trends related to complex creep deformation have been generated to recognize phase transitions in critical phenomena. In mechanics, creep is the affinity of a solid material to move bit by bit or deform everlastingly influenced by mechanical forces. It can occur on the account of long term exposure to increased pressure that are yet below the influences of the yield nature of the material. Creep is progressively extreme in materials exposed to high loads and heat for long time periods. Shell structures have many designing applications like avionics, rocketry, electric engines, and train motors. Researchers have discovered its expanding application in the field of civil and mechanical engineering, /1/. These spherical structures are powerful from the viewpoints of both structural and building plan. In many of these cases, spherical shells have to operate under severe mechanical and thermal loads. The breakdown or damage is started by jerk, pressure, and load impacts, or from their influence on structures that both experience and do not encounter natural hazards. Therefore, interest in fortifying and updating current solid structures, in view of the harm brought about by long haul impacts and unreasonable basic distortions has been perceived, /2/. The nonlinear long term creep conduct of the spherical shell, meagre walled solid shells of transformation (including domes) exposed to continued loads are researched by Hamed, /3/. A nonlinear axisymmetric hypothetical model, representing the impacts of creep and shrinkage, and which concerns aging of the solid material and the variety of interior variables and geometry in time, is created for this reason. Miller /4/ has worked on the nature of stresses and displacements in a compressible circular shell exposed to inner and outer pressure. Carrera et al. /5/ discussed various effects in shell structures due to anisotropic behaviour and layered structures, such as high transverse deformability, zig-zag effects, etc. Contributions based on axiomatic, asymptotic, and continuum based approaches have been overviewed. Tornabene et al. /6/ studied the dynamic behaviour of functionally graded parabolic and circular panels and shells of revolution. The first-order shear deformation theory is used to study these moderately thick structural elements. Yakovlev et al. /7/ presented a mathematical model for isothermal deformation of dome-shaped shells made of high-strength anisotropic materials in the presence of creep. The model is used to assess the flow kinetics, the stress-strain state, the forces, sizes of manufactured parts, and limiting deformation. Verma et al. /8/ established the mathematical model on elastic-plastic transitions occurring in rotating spherical shells, based on compressibility of materials. Stresses are computed for initial yielding and fully plastic state of rotating spherical shell. In this paper, creep behaviour of the shell is characterised by taking strain components in general form and Seth's concept of measure. Creep progress in structures has been effectively connected to an extensive number of problems, /9-15/.

Seth /16, 17/ has characterised the idea of generalised strain measures as:

$$e_{ii} = \int_0^{e_{ii}^A} \left[1-2e_{ii}^A\right]^{\frac{n}{2}-1} d\,e_{ii}^A = \frac{1}{n}\left[1-\left(1-2e_{ii}^A\right)^{\frac{n}{2}}\right], \; (i = 1,2,3), \quad (1)$$

where: n deals with measure; and e^A_{ii} are the Almansi finite strain components.

FORMULATION OF THE MATHEMATICAL MODEL

Think about a spherical shell whose interior and outer radii are A and B, that is exposed to uniform inner pressure p_i of gradually increasing to the inside shell surface $r = A$.

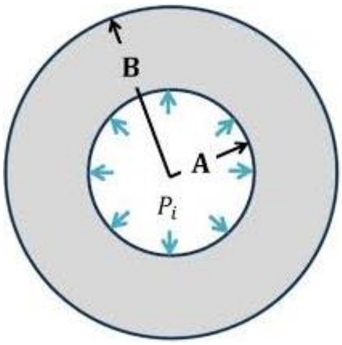

Figure 1. Mathematical model of shell under pressure.

DISPLACEMENT COORDINATES

Let X, Y, Z are displacement coordinates of a particle in the shell under the influence of pressure given as spherical coordinates

$$X = r(1-\beta), \; Y = 0, \; Z = 0, \quad (2)$$

where: X, Y, Z (displacement components); β is position function, contingent upon r.

In finite elasticity, Almansi has developed measures in which deformed and undeformed states are taken as reference frameworks respectively extensively used. Almansi summed up the segments of strain and showed them by using weighted integral measures known as generalised strain measure. In this problem, the components of strain are taken in generalised form as

$$e_{rr} = \frac{1}{n}[1-(r\beta'+\beta)^n], \quad e_{\theta\theta} = \frac{1}{n}[1-\beta^n] = e_{\phi\phi},$$

$$e_{r\theta} = e_{\theta\phi} = e_{\phi r} = 0, \quad (3)$$

where: n is measure; and $\beta' = d\beta/dr$.

STRESS-STRAIN RELATIONSHIP

Sokolnikoff /18/ derived the stress-strain relationship for isotropic material given as

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij}, \; (i, j = 1, 2, 3), \quad (4)$$

where symbols have their usual meaning.

Substituting the strain components from Eq.(3) in Eq.(4), the stresses are obtained as:

$$T_{rr} = \frac{2\mu}{n}[1-(r\beta'+\beta)^n] + \frac{\lambda}{n}[3-(r\beta'+\beta)^n - 2\beta^n],$$

$$T_{\theta\theta} = T_{\phi\phi} = \frac{2\mu}{n}[1-\beta^n] + \frac{\lambda}{n}[3-(r\beta'+\beta)^n - 2\beta^n],$$

$$T_{r\theta} = T_{\theta\phi} = T_{\phi r} = 0. \quad (5)$$

EQUILIBRIUM EQUATION

The equilibrium condition of the spherical shell satisfying the following condition /19/, is:

$$\frac{dT_{rr}}{dr} + \frac{2}{r}(T_{rr} - T_{\theta\theta}) = 0, \quad (6)$$

where: T_{rr} and $T_{\theta\theta}$ are the radial and hoop stresses, respectively.

ASYMPTOTIC SOLUTION AT TRANSITION POINTS

Using Eq.(5) in Eq.(6), we get a nonlinear differential equation in β as:

$$n(2-C)P(P+1)^{n-1}\beta^{n+1}\frac{dP}{d\beta} = \Big[\beta^n\{2-(1+P)^n\} -$$

$$-n\beta^n P\{(1-C)+(2-C)(1+P)^n\}\Big], \quad (7)$$

where: $C = 2\mu/\lambda + 2\mu$; and $r\beta' = \beta P$ (P is a function of β and β is a function of r only).

The transition points or turning point of β in Eq.(7) are $P \to -1$ and $P \to \pm\infty$.

The initial condition of the pressure in the spherical shell is described as:

$$T_{rr} = -p_i \quad \text{at} \quad r = A,$$

$$T_{rr} = 0 \quad \text{at} \quad r = B. \quad (8)$$

THEORETICAL SOLUTION OF THE MATHEMATICAL MODEL

The solution of the problem means to calculate the creep stress and strain rates in the spherical shell under the influence of pressure. In order to obtain the solution, we set a transition function of the problem by taking the difference through principal stresses at transition point $P \to -1$, /20-27/. We characterize the transition function T_f as:

$$T_f = T_{rr} - T_{\theta\theta} = \frac{2\mu\beta^n}{n}[1-(P+1)^n], \quad (9)$$

where: $T_f = f(r)$.

Taking the logarithmic function on both sides of Eq.(9) and differentiating Eq.(9) with respect to β and by using the value of $dP/d\beta$ from Eq.(7), we get:

$$\frac{d}{d\beta}(\log T_f) = \frac{np}{r[1-(P+1)^n]}\left[1-(1+P)^n - \beta(P+1)^n\frac{dP}{d\beta}\right], \quad (10)$$

$$\frac{d}{d\beta}\log T_f = -\frac{[2C(1-2n)+3n]}{\beta}. \quad (11)$$

Integrating Eq.(11) with respect to β, we get

$$T_f = T_{rr} - T_{\theta\theta} = k_1 r^q, \quad (12)$$

where: $$q = -[3n + 2C(1-2n)], \quad (13)$$

and k_1 is an integration constant.

By using Eq.(12) in Eq.(6), we get

$$T_{rr} = -2k_1\int r^{q-1}dr + k_2, \quad (14)$$

where: k_2 is an integration constant.

Using initial condition (i) in Eq.(14), we get

$$k_2 = 2k_1 \int_{r=B} r^{q-1} dr \,.$$

By taking the value of k_2 in Eq.(14), we get

$$T_{rr} = 2k_1 \int_r^B r^{q-1} dr \,. \tag{15}$$

By using Eq.(15) in Eq.(12), we get

$$T_{\theta\theta} = 2k_1 \left[\int_r^B r^{q-1} dr - \frac{r^q}{2} \right]. \tag{16}$$

Using initial condition (ii) in Eq.(16), we get

$$k_1 = -\frac{p}{2\int_A^B r^{q-1} dr} \,.$$

By taking the value of k_1 in Eqs.(15) and (16), we get radial and hoop (circumferential) stresses as

$$T_{rr} = -p \frac{\int_r^B r^{q-1} dr}{2\int_A^B r^{q-1} dr}, \quad T_{\theta\theta} = T_{rr} + \frac{pr^q}{2\int_A^B r^{q-1} dr} \,. \tag{17}$$

It is noticed that estimation of $|T_{\theta\theta} - T_{rr}|$ is greatest at $r = A$. Hence, the underlying yielding begins at the interior surface of the shell given as

$$Y = T_{\theta\theta} - T_{rr} = \frac{pA^q}{2\int_A^B r^{q-1} dr} \,, \tag{18}$$

where: Y is yield stress.

Therefore, initial pressure required to start yielding in the spherical shell is given as

$$P_i = \frac{p}{Y} = \frac{2}{q}\left(\left(\frac{B}{A} \right)^q - 1 \right). \tag{19}$$

Equations (17) give creep stresses for the spherical shell under the effect of pressure. By using the following non-dimensional components: $M = r/B$, $M_0 = A/B$, $S_r = T_{rr}/p$, $S_\theta = T_{\theta\theta}/p$, new stress equations in a non-dimensional form are

$$S_r = -\frac{\int_M^1 M^{q-1} dM}{2\int_{M_0}^1 M^{q-1} dM} \,, \tag{20}$$

$$S_\theta = S_r + \frac{M^q}{2\int_{M_0}^M M^{q-1} dM} \,. \tag{21}$$

Fully plastic state

As a specific case, we get the transitional creep stresses for completely plastic condition of the spherical shell by C approaching nearer to zero. Along these lines, conditions Eqs.(20)-(21) become

$$S_{rf} = -\frac{\int_M^1 M^{q_1-1} dM}{2\int_{M_0}^1 M^{q_1-1} dM} \,, \tag{22}$$

$$S_{\theta f} = S_{rf} + \frac{M^{q_1}}{2\int_{M_0}^M M^{q_1-1} dM} \,, \tag{23}$$

where: $q_1 = -3n$.

ESTIMATION OF CREEP PARAMETERS

Further, the strain rates can be determined for this creep investigation of the spherical shell under interior pressure. For this reason, the creep set in the strain ought to be replaced by creep strain rates and the stress-strain relationship. Equations (4) become

$$\dot{e}_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij} T \,, \tag{24}$$

where: $\dot{e}_{ij}$ is strain rate tensor with respect to flow parameter t. Differentiating the second Eq.(3) with respect to time t, we get

$$\dot{e}_{\theta\theta} = -\beta^{n-1}\dot{\beta} \,. \tag{25}$$

For Swainger measure (i.e., $n = 1$), Eq.(25) becomes

$$\dot{\varepsilon}_{\theta\theta} = \dot{\beta} \,, \tag{26}$$

where: $\dot{\varepsilon}_{\theta\theta}$ is the Swainger strain measure. From Eq.(9) the transition value β is given by

$$\beta = (n/2\mu)^{1/n} [S_r - S_\theta]^{1/n} \,. \tag{27}$$

Using Eqs.(25)-(27) in Eq.(24), we get

$$\begin{aligned} \dot{\varepsilon}_{rr} &= k[S_r - \nu(S_\theta + S_\phi)] \,, \\ \dot{\varepsilon}_{\theta\theta} &= k[S_\theta - \nu(S_r + S_\phi)] \,, \\ \dot{\varepsilon}_{\phi\phi} &= k[S_\phi - \nu(S_\theta + S_r)] \,, \end{aligned} \tag{28}$$

where: $k = \left[\frac{n}{2\mu}(S_r - S_\theta)E \right]^{\frac{1}{n}}$; and $\dot{\varepsilon}_{rr}$, $\dot{\varepsilon}_{\theta\theta}$, and $\dot{\varepsilon}_{\phi\phi}$ are strain rates tensor.

These are the constitutive conditions utilised by Odquist /28/ for finding the creep stress and strain rates when we put $n = 1/N$.

NUMERICAL DISCUSSION ON RESULTS

For computing creep stresses and strain rates for the above mathematical model, the accompanying qualities have been taken, compressible materials rubber, clay, concrete, and measures $n = 1/3, 1/5, 1/7$ (for example $N = 3, 5, 7$). In classical mechanics, measure N is equivalent to $1/n$. The definite integrals in Eqs.(20)-(21) are assessed by taking Simpson's rule.

To see the effect of creep stresses and strains on a spherical shell, the following values have been taken $C = 0$ (e.g., rubber), 0.25 (clay), 0.75 (concrete), and are shown in Fig. 2. Curves are plotted between creep stresses along the radii proportion $M = r/B$ (see Fig. 2) for the spherical shell made up of different material having compressibility. It is seen from Fig. 2 that the σ_θ (hoop stress) has greatest effect at the outer surface of the spherical shell when contrasted with the σ_r (radial stress). For measure $n = 1/3$, it has been seen that the hoop stress has a maximum value at the internal surface of the shell for $C = 0.75$ (compressible material) and values decrease with decrease in compressibility of the material. It means that the spherical shell is under maximal influence of creep stresses for measure $n = 1/3$ as compared to $n = 1/5, 1/7$.

In Fig. 3, strain rates are plotted against $M = r/b$ for measure $n = 1/3, 1/5, 1/7$, and notations *err* and *eqq* in Fig. 3 represent strain rates $\dot{\varepsilon}_{rr}$ and $\dot{\varepsilon}_{\theta\theta}$, respectively. It is seen that the strain rates have the greatest value at the internal surface of the spherical shell for all values of measure. It is also observed that *eqq* has more effect on the inner surface of the spherical shell as compared to the *err*. The values of strain rates are maximum for compressible material $C = 0.75$

for all values of the measure, and the effect of strain rates decreases with decrease in compressibility of the material.

Pressure required to initiate creep behaviour in the spherical shell made up of different thickness ratio is studied in Fig. 4. Pressure is calculated for different types of material with compressibility for measure n = 1/3, 1/5, 1.7. It is seen that high pressure is required to generate creep in spherical shell for measure n = 1/7 as compared to n = 1/3, 1/5. This indicates that creep phenomenon starts at low value of pressure in the spherical shell for n = 1/3 as compared to value n = 1/7, 1/5. It means that the possibility of damage of the spherical shell is maximum at the internal surface for n = 1/3. It is also seen that thinner spherical shells yield at lower values of pressure as compared to thicker shells. Pressure is calculated for fully plastic state in Fig. 5. These pressure values are increased as compared to initial state of creep.

Nomenclature

A	inner radius of spherical shell
B	outer radius of spherical shell
λ, μ	Lame's constants
δ_{ij}	Kronecker's delta
e_{ii}	strain invariant
T_{ii}	stress invariant
e^A_{ij}	principal Almansi finite strain components
n	measure
p	pressure
u, v, w	displacement components
r, θ, ϕ	spherical coordinates
β	function of r only
P	function of β only
T_f	transition function
Y	yield stress
ν	Poisson's ratio
$M_0 = a/b$	radii ratio
σ_{rr}	radial stress component

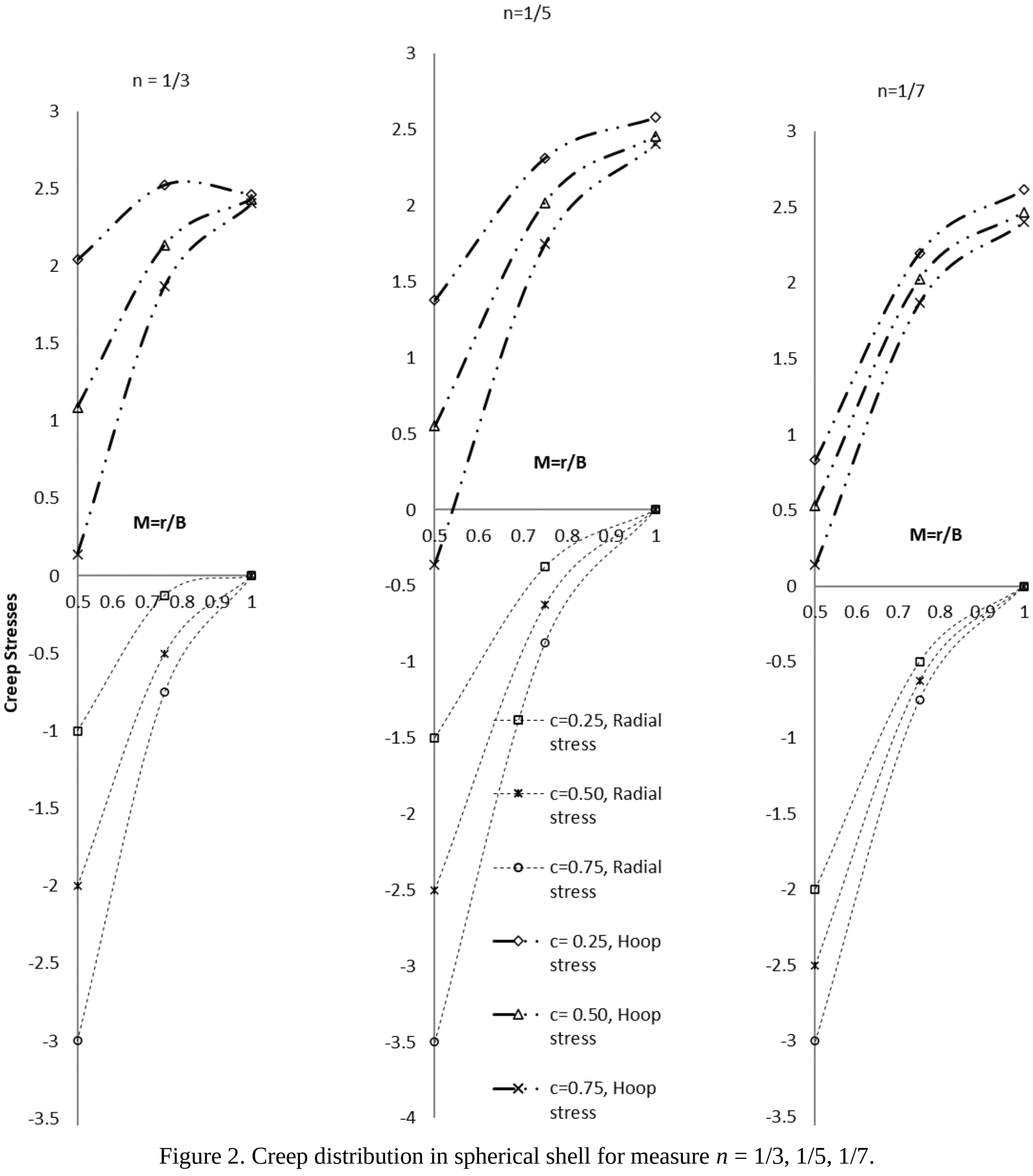

Figure 2. Creep distribution in spherical shell for measure n = 1/3, 1/5, 1/7.

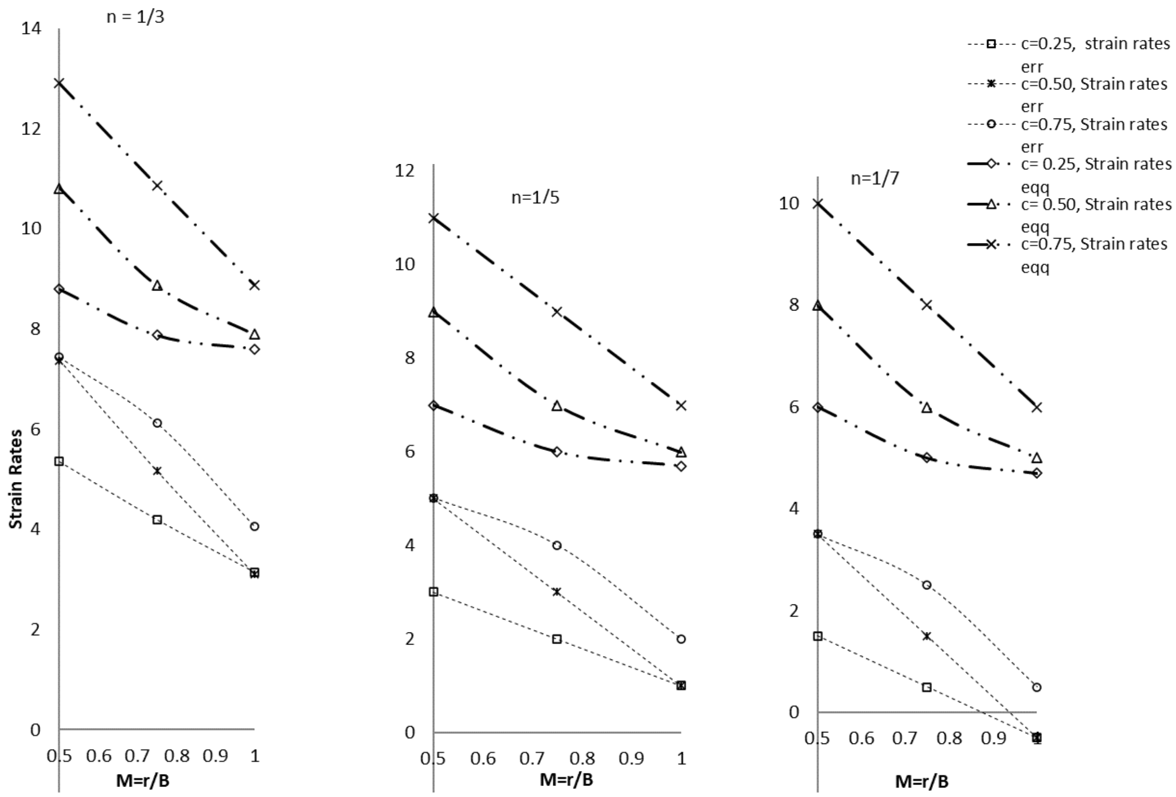

Figure 3. Creep strain rates in spherical shell for measure *n* = 1/3, 1/5, 1/7.

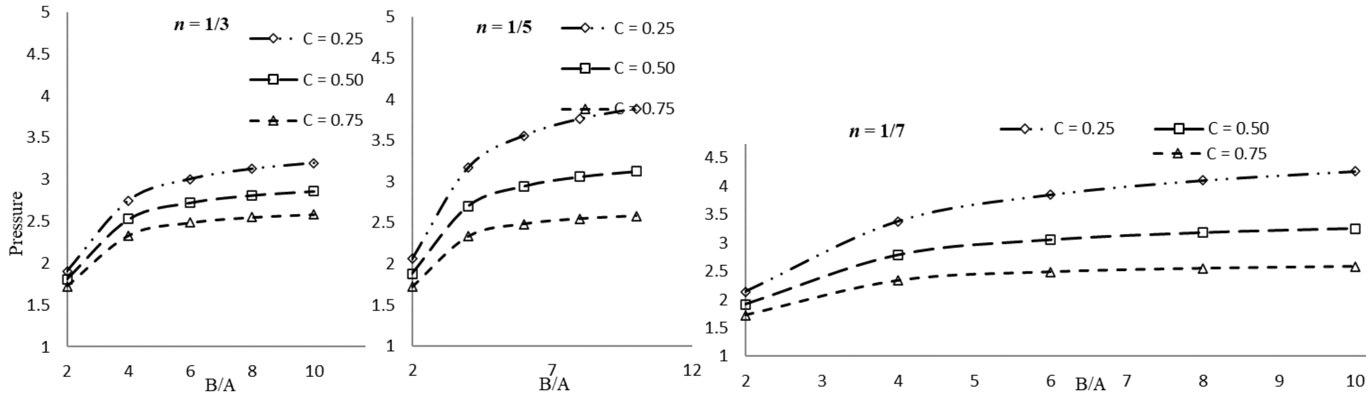

Figure 4. Pressure required for onset of yielding in spherical shell for measure *n* = 1/3, 1/5, 1/7.

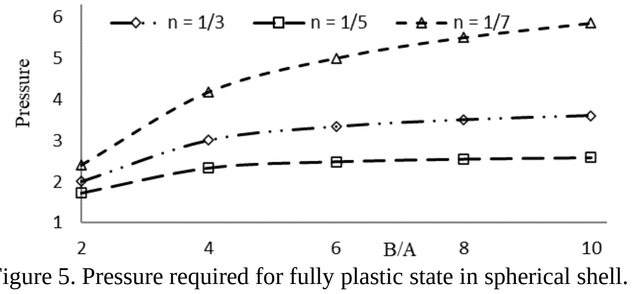

Figure 5. Pressure required for fully plastic state in spherical shell.

REFERENCES

1. Shambharkar, R., *Vibration analysis of thin rotating cylindrical shell*, Master Thesis, National Institute of Technology Rourkela, Orissa, India, 2008.
2. Thakur, P., Singh, S.B., Pathania, D., Verma, G. (2017), *Thermal creep stress and strain analysis in non-homogeneous spherical shell*, J Theor. Appl. Mech. 55(4): 1155-1165. doi: 10.15632/jtam-pl.55.4.1155
3. Hamed, E., Bradford, M.A., Gilbert, R.I. (2010), *Nonlinear long-term behaviour of spherical shallow thin-walled concrete shells of revolution*, Int. J Solids Struct. 47(2): 204-215. doi: 10.1016/j.ijsolstr.2009.09.027
4. Miller, G.K. (1995), *Stresses in a spherical pressure vessel undergoing creep and dimensional changes*, Int. J Solids Struct. 32(14): 2077-2093. doi: 10.1016/0020-7683(94)00197-5
5. Carrera, E. (2002), *Theories and finite elements for multi-layered, anisotropic, composite plates and shells*, Arch. Comput. Meth. Eng. 9(2): 87-140. doi: 10.1007/BF02736649
6. Tornabene, F., Viola, E. (2009), *Free vibration analysis of functionally graded panels and shells of revolution*, Meccanica, 44 (3): 255-281. doi: 10.1007/s11012-008-9167-x
7. Yakovlev, S.S., Larin, S.N., Sobolev, Y.A., Platonov, V.I. (2015), *Isothermal deformation of dome-shaped shells made of high-strength anisotropic materials in the presence of creep*, Russ. Engin. Res. 35(2):116-120. doi: 10.3103/S1068798X15020240
8. Thakur, P., Verma, G., Pathania, D.S., Satya, B.S. (2017), *Elastic-plastic transition on rotating spherical shells in dependence*

of compressibility, Kragujevac J Sci. 39: 5-16. doi: 10.5937/kgjsci1739005t
9. Verma, G., Rana, P., Pathania, D.S., Thakur, P. (2017), *Creep transition in the rotating spherical shell under the effect of density variable by Seth's transition theory*, AIP Conference proceeding 1802(1), 020020 (2017). doi: 10.1063/1.4973270
10. Verma, G., Rana, P. (2017), *Creep stresses in a spherical shell under steady state temperature*, AIP Conf. Proc. 1897(1): 020033. doi: 10.1063/1.5008712
11. Thakur, P. (2014), *Steady thermal stress and strain rates in a circular cylinder with non-homogeneous compressibility subjected to thermal load*, Therm. Sci. 18(1): 81-92. doi: 10.2298/TSCI110315080P
12. Thakur, P., Kaur, J., Singh, S.B. (2016), *Thermal creep transition stresses and strain rates in a circular disc with shaft having variable density*, Eng. Comput. 33(3). doi: 10.1108/EC-05-2015-0110
13. Pathania, D.S., Verma, G. (2019), *Temperature and pressure dependent creep stress analysis of spherical shell*, Int. J Appl. Mech. Eng. 24(1): 105-115. doi: 10.2478/ijame-2019-0007
14. Verma, G., Thakur, P., Rana, P. (2019), *Elastic-plastic analysis of transversely isotropic spherical shell under internal pressure*, AIP Conf. Proc. 2061(1): 020031. doi: 10.1063/1.5086653
15. Verma, G. (2017), *Effect of external pressure on spherical shell*, RC, Int. J Eng. Sci. 24: 61-73.
16. Seth, B.R. (1962), *Transition theory of elastic-plastic deformation, creep and relaxation*, Nature, 195: 896-897. doi: 10.1038/195896a0
17. Seth, B.R. (1966), *Measure-concept in mechanics*, Int. J Non-Linear Mech. 1(1): 35-40. doi: 10.1016/0020-7462(66)90016-3
18. Sokolnikoff, I.S., Mathematical Theory of Elasticity, Second Ed., McGraw-Hill Book Co., New York, 1956.
19. Gupta, S., Verma, G. (2014), *Elastic plastic transition in shells under internal pressure*, Int. J Emerg. Techn. & Adv. Eng. 4(8): 126-129.
20. Thakur, P., Pathania, D.S., Verma, G., Singh, S.B. (2017), *Elastic-plastic stress analysis in a spherical shell under internal pressure and steady state temperature*, Struct. Integr. Life, 17(1): 39-43.
21. Gupta, S., Verma, G. (2015), *Creep transition of spherical shell under internal pressure*, ISJ Theor. Appl. Sci. 24(4): 201-207. doi: 10.15863/TAS.2015.04.24.35
22. Verma, G. (2020), *Creep modelling of spherical shell under influence of internal and external pressure*, Struct. Integr. Life, 20(2): 93-97.
23. Verma, G., Pathania, D.S. (2020), *Thermal creep stress analysis of functionally graded spherical shell under internal and external pressure*, Struct. Integr. Life, 20(3): 297-302.
24. Verma, G., Thakur, P. (2022), *Comparative creep analysis of spherical shell made up of different materials*, Mech. Solids, 57 (5): 1214-1221. doi: 10.3103/S0025654422050120
25. Kumar, A., Sood, S., Thakur, P., Kumar N. (2022), *Safety analysis in an isotropic material disc fitted with a rigid shaft subjected to thermo-mechanical load by using transition theory*, Struct. Integr. Life, 22(2): 193-201.
26. Kumar N., Thakur, P. (2022), *Behaviour of creep stresses and strain rates in a solid disc made of isotropic material with thermal condition*, Struct. Integr. Life, 22(2): 202-209.
27. Verma, G., Thakur, P., Radaković, Z. (2022), *Comparative study of pressurized spherical shell with different materials*, In: Gupta, G., Wang, L., Yadav, A., Rana, P., Wang, Z. (eds) Proc. of Academia-Industry Consortium for Data Science. Advances in Intelligent Systems and Computing, vol 1411. Springer, Singapore. doi: 10.1007/978-981-16-6887-6_18
28. Odqvist, F.K.G., Mathematical Theory of Creep and Creep Rupture, 2nd Ed., Clarendon Press, Oxford, USA, 1974.