Gaurav Verma[1], Gagandeep[1], Ishika[2], Hargun Kaur Sodhi[2]*

DEFORMATION ANALYSIS OF SPHERICAL SHELL BY USING GENERALISED HOOKE'S LAW

ANALIZA DEFORMACIJA SFERNE LJUSKE PRIMENOM OPŠTEG HUKOVOG ZAKONA

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Adresa autora / Author's address:
[1)] Department of Mathematics, Hans Raj Mahila Maha Vidyalaya, Jalandhar, Punjab, India
[*]email: gkdon85@gmail.com
[2)] BSc Student, Hans Raj Mahila Maha Vidyalaya, Jalandhar, Punjab, India

Keywords

- deformation
- spherical shell
- transition

Abstract

Deformation analysis of spherical shell structures is significant for safety measure and structural design optimisation. Spherical shells are frequently used as important components in mechanical, aerospace, and civil engineering. In this paper, generalised Hooke's law is taken to analyse the elastic-plastic deformations in shell-type structures. The problem of spherical shell is formulated and solved with the help of generalised strain measures and transition theory. Mathematical results are derived to show stress behaviour in the deformed spherical shell. Results obtained are shown graphically.

Ključne reči

- deformacija
- sferna ljuska
- prelazni naponi

Izvod

Analiza deformacija u konstrukcijama tipa sferne ljuske su značajne za proveru sigurnosti i radi optimizacije u projektovanju. Sferne ljuske se često koriste kao važne komponente u mehaničkim, kosmičkim i građevinskim konstrukcijama. U ovom radu se primenjuje opšti Hukov zakon za analizu elastoplastičnih deformacija u konstrukcijama tipa ljuske. Data je formulacija problema sferne ljuske sa rešenjem, primenom generalisanih mera deformacije teorije prelaznih napona. Rezultati su matematički izvedeni radi predstavljanja ponašanja napona u deformisanoj sfernoj ljuski. Dobijeni rezultati su prikazani i grafički.

INTRODUCTION

The advanced research in mechanics deals with elastic-plastic deformations which occur under stress and strain relations. Transition is a natural phenomenon and there is hardly any branch of science and technology in which we do not come across transition from one state to another. The fundamental structure of the medium undergoes various changes /1/ during transitions. Deformation plays a vital role in the transitions as compression, deflection, or extension result in variations in shape, size, or position of the body /2/. A deformation may occur due to body forces, internal pressure, external load, or temperature changes in the body. Deformations which are recovered after the removal of body forces are called elastic. In this case, a body completely recovers its original configuration. On the other hand, if deformation remains, even after body forces are removed, it is called plastic deformation. It is one of the types of irreversible deformations in which material bodies, after stresses, have attained a certain threshold value known as elastic limit or yield stress. If the state of deformation in a body remains constant throughout the whole part of the material body, then it is called homogeneous deformation. Thus, the deformation is responsible for change in the body behaviour from elastic to plastic and plastic to body fracture. In the current literature, the study of elastic-plastic deformation is done by the concept of generalised measures and Seth's transition theory. Both types of these deformations can be dependent or independent of time. The tendency of a solid material to move slowly or deform permanently occurs as result of long-term exposure to high level of stress or creep deformation /3/. Creep deformation does not occur suddenly upon the application of stress.

OBJECTIVES OF STUDY

- To develop the knowledge, skills, and attitudes necessary to pursue research activities in mathematics.
- To develop a mathematical model to analyse a deformation of spherical shells for sustainability.
- To analyse the behaviour of elastic-plastic deformations in a spherical shell.

NATURE OF THE ELASTIC-PLASTIC TRANSITION

In 1868, Tresca considered that there exists a mid-zone area between elastic and plastic states of solid which is against the classical theory of elastic-plastic transition. The classical model talks only about the linear behaviour of elastic-plastic transition as shown in Fig. 2, but this model does not talk about the nonlinear behaviour of transition as shown in Fig. 3, /4/. Many authors have not recognised this intermediate state as a separate state, that of elastic and plastic. It was first B.R. Seth who explained the nonlinear nature of transition state. Therefore, B.R. Seth has elaborated the concept of this intermediate region. He has named this region as 'transition region', as shown in Fig. 3. He has developed a transition theory of elastic-plastic and creep transition. Seth /5/ has defined the concept of generalised strain measure which, when applied to the governing differential equation of the medium, eliminates ad-hoc assumptions as incompressibility, strain law, yield condition, etc., and these constitutive equations give elastic-plastic and creep results through some transition functions.

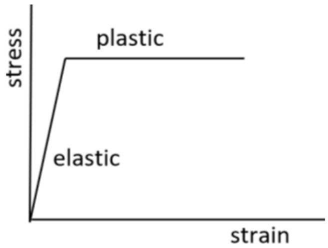

Figure 1. Elastic-plastic deformation without transition.

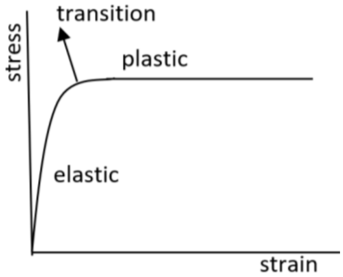

Figure 2. Elastic-plastic deformation with transition.

MATHEMATICAL MODEL OF ELASTIC-PLASTIC DEFORMATION IN THE SPHERICAL SHELL

Spherical shell

A spherical shell is a curved surface in which the thickness is much smaller compared to other dimensions. The geometrical properties of a spherical shell, i.e., single, or double curvature give rise to a tremendous advantage of these lightweight structures. Elastic-plastic and creep deformation of spherical-shell structures have an important role in engineering applications. Engineers have found increasing applications in aerospace, civil, and mechanical industries, chemical such as in high-speed centrifugal separators, gas turbines for high-power aircraft engines, spinning satellite structures, certain rotor systems, and rotating magnetic shields /6/. To increase the life and strength of spherical shells, it is therefore very important for engineers to study the behaviour of elastic-plastic and creep deformation of spherical shells.

Literature review

These spherical structures are powerful from the viewpoints of both structural and building plan. In many of these cases, the spherical shells have to operate under severe mechanical and thermal loads. The breakdown or damage is started by jerk, pressure, and load impacts, or from their communication with structures that both experience and do not encounter natural malfunction. Therefore, interest for fortifying and updating current solid structures, in view of harm brought about by long haul impacts and unreasonable basic distortions has been perceived /7/. The nonlinear long term creep conduct of spherical shell, meagre walled solid shells of transformation (including domes) exposed to continuous loads is researched by Hamed et al. /8/. A nonlinear axisymmetric hypothetical model that represents the impacts of creep and shrinkage with concerns of solid material aging and a variety of internal effects and geometry in time, is created for this reason. Miller et al. /9/ have worked on the nature of stresses and displacements in a compressible circular shell exposed to inner and outside pressure. Carrera et al. /10/ discussed various effects in shell structures due to anisotropic behaviour and layered constructions, such as high transverse deformability, zig-zag effects, etc. Contributions based on axiomatic, asymptotic, and continuum-based approach have been overviewed. Tornabene et al. /11/ studied the dynamic behaviour of functionally graded parabolic and circular panels and shells of revolution. The first-order shear deformation theory is used to study these moderately thick structural elements. Yakovlev et al. /12/ presented a mathematical model for the isothermal deformation of dome-shaped shells made of high-strength anisotropic material in the presence of creep. The model is used to assess the flow kinetics, the stress-strain state, the forces, the dimensions of manufactured parts, and the limiting deformation. Verma et al. /13/ established the mathematical model on elastic-plastic transitions occurring in rotating spherical shells based on compressibility of materials. Stresses are computed for initial yield and fully plastic state of the rotating spherical shell. Deformation behaviour of the shell is characterised by taking strain components in general form and Seth's concept of measure. The elastic-plastic and creep progress in structures has been effectively connected to an extensive number of problems, /14-20/.

Formulation of the problem

Here in current literature, we discuss the research problem of elastic-plastic deformation in shells under internal pressure. Consider a spherical shell of constant thickness under internal pressure. Due to the symmetry in elastic properties, the displacement is purely radial. Therefore, we take the displacements in spherical coordinates (r, θ, ϕ) as

$$X = r(1-\alpha),\ \ Y=0,\ \ Z=0 \tag{1}$$

where: α is function of $r = \sqrt{(x^2 + y^2 + z^2)}$.

The generalised components of strain are defined as

$$e^A_{rr} = \frac{1}{2}[1-(r\alpha'+\alpha)^n]\,,$$
$$e^A_{\theta\theta} = e^A_{\varphi\varphi} = \frac{1}{2}(1-\alpha^n)\,, \tag{2}$$
$$e^A_{r\theta} = e^A_{\theta\varphi} = e^A_{r\varphi} = 0\,,$$

where: $\alpha' = d\alpha/dr$.

GENERALISED HOOKE'S LAW

The stress-strain relations for solid material in generalised from are given by /21/

$$T_{ij} - \lambda\delta_{ij} I_1 = 2\mu e_{ij}\,,\quad (i,j=1,2,3)\,, \tag{3}$$

where: λ and μ are Lame's constants; and $I_1 = e_{kk}$ is called first strain invariant.

Using Eqs.(2) in (3), the corresponding stresses are given

$$T_{rr} = \frac{\lambda+2\mu}{n}[1-(r\alpha'+\alpha)^n] + \frac{2\lambda}{n}(1-\alpha^n)\,,$$
$$T_{\theta\theta} = T_{\varphi\varphi} = \frac{\lambda}{n}[1-(r\alpha'+\alpha)^n] + \frac{2\lambda+2\mu}{n}(1-\alpha^n)\,, \tag{4}$$
$$T_{r\theta} = T_{\theta\varphi} = T_{r\varphi} = 0\,.$$

The equations of equilibrium are all satisfied except

$$r\frac{dT_{rr}}{dr}+2(T_{rr}-T_{\theta\theta})=0\,. \qquad (5)$$

Using Eq.(4) in Eq.(5), we get the nonlinear differential equation in β as

$$\frac{d\alpha}{dM}\left[(1+M)^n+2(1-C)-\frac{2C(1-(1+M)^n)}{nM}\right]+\alpha(1+M)^{n-1}=0 \quad (6)$$

where: $C = 2\mu/\lambda + 2\mu = (1 - 2\sigma)/(1 - \sigma)$; $r\alpha' = \alpha M$.

Turning points of α from Eq.(6) are $M \to 1$ and $M \to \pm\infty$.

Boundary conditions of the problem are given as

$$T_{rr} = -p \text{ at } r = a, \text{ and } T_{rr} = 0 \text{ at } r = b.$$

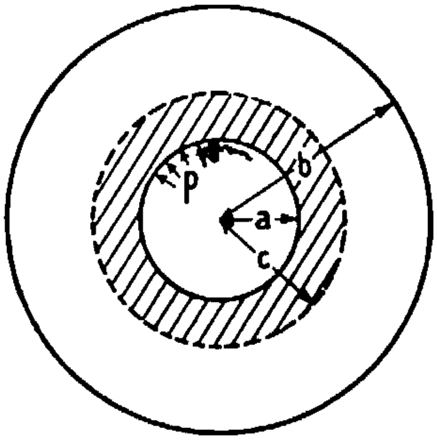

Figure 3. Expansion of plastic region with internal pressure, /6/.

Further, in order to find the plastic stress, the transition function is defined using the principal stress as taken by Verma /22-27/ at the transition point $M \to \pm\infty$. The transition function is defined as

$$\Psi=1-\frac{nT_{rr}}{3\lambda+2\mu}=\frac{\alpha^n}{3-2C}[(1+M)^n+2(1-C)]\,. \qquad (7)$$

Taking the logarithmic differentiation of Eq.(7) with respect to r and using Eq.(6), we get

$$\frac{d\log\Psi}{dr}=\frac{2C[1-(1+M)^n]}{r[(1+M)^n+2(1-C)]}\,. \qquad (8)$$

Taking the asymptotic value of Eq.(8) at $M \to \pm\infty$ and integrating, we get

$$\Psi=A_0 r^{-2C}\,, \qquad (9)$$

where: A_0 is a constant of integration which can be determined by boundary condition. From Eq.(7), we have the transition value of T_{rr} as

$$T_{rr}=\frac{2\mu(3-2C)}{nC}[1-A_0 r^{-2C}]\,. \qquad (10)$$

The value of E in the transition range is given by Seth /8/

$$Y=\frac{E}{n}=\frac{2\mu(1+\sigma)}{n}=\frac{2\mu(3-2C)}{n(2-C)}\,. \qquad (11)$$

Using Eq.(11) in Eq.(10), we have

$$T_{rr}=\frac{Y(2-C)}{C}[1-A_0 r^{-2C}]\,. \qquad (12)$$

Applying boundary condition, $T_{rr} = 0$ at $r = b$ and using Eq.(5)

$$T_{rr}=\frac{Y(2-C)}{C}\left[1-\left(\frac{b}{c}\right)^{2C}\right],$$

$$T_{\theta\theta}-T_{rr}=Y(2-C)(b/r)^{2C}\,. \qquad (13)$$

For initial yielding, $|T_{\theta\theta} - T_{rr}|$ is maximum at $r = a$. Therefore, yielding will start at the internal surface of the sphere,

$$\left|T_{\theta\theta}-T_{rr}\right|_{r=a}=Y(2-C)(b/a)^{2C}\,.$$

By boundary condition $T_{rr} = -p$ at $r = a$ in Eq.(13). Therefore, pressure required to start initial yielding is

$$p_i=\frac{Y(2-C)}{C}\left[\left(\frac{b}{a}\right)^{2C}-1\right]. \qquad (14)$$

Non-dimensional components are introduced in Eqs.(13) and (14) as

$R = r/b$, $R_0 = a/b$, $S_{rr} = T_{rr}/Y$, $S_{\theta\theta}\, T_{\theta\theta}/Y$, $P = p_i/Y$,

$S_{rr} = \{(2 - C)/C\}[1 - (R)^{-2C}]$, $S_{\theta\theta} = S_{rr} + (2 - C)(R)^{-2C}$, (15)

$$P=\frac{(2-C)}{C}[(R_0)^{-2C}-1]\,. \qquad (16)$$

Results obtained in Eqs.(15) and (16) are non-dimensional components.

RESULTS AND DISCUSSION

For calculating stresses and pressure based on the above mathematical analysis of the shell problem, the values of compressibility factor are taken as $C = 0.63$ (isotropic material steel), $C = 0.48$ (isotropic material copper), and $C = 0.25$ (isotropic material polypropylene). In Fig. 4, graphical analysis between non-dimensional stress distribution (i.e., S_{rr}, $S_{\theta\theta}$) vs. radii ratio $R = r/b$ for the spherical shell made of different materials is done. Stress behaviour of deformed spherical shell is seen along the radii ratio of the shell. Values of radial stresses are found to be negative for all three types of material due to the boundary condition, whereas circumferential stresses increase along radii ratio $R = r/b$. The circumferential stresses have maximal impact on the external part of spherical shell for polypropylene material as compared to steel and copper. In Fig. 5, the pressure required to start yielding is computed along radii ratio $R = r/b$. It is observed that pressure required to start yielding in the shell is maximum for steel and copper material as compared to polypropylene. It means that the spherical shell made of polypro

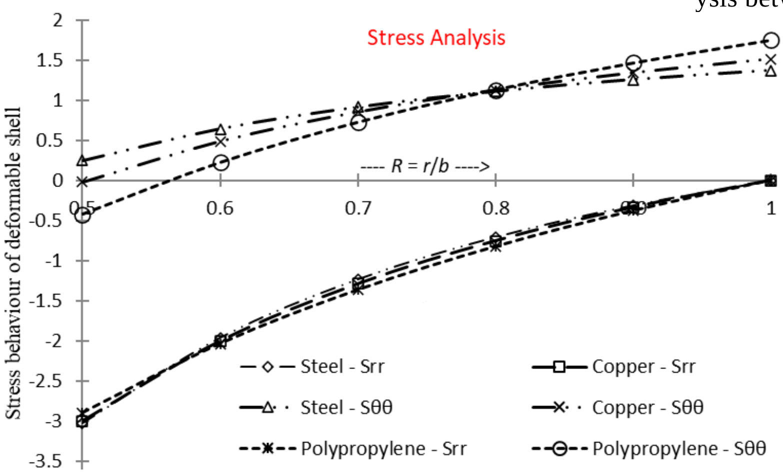

Figure 4. Stress behaviour of deformed spherical shell along radii ratio $R = r/b$.

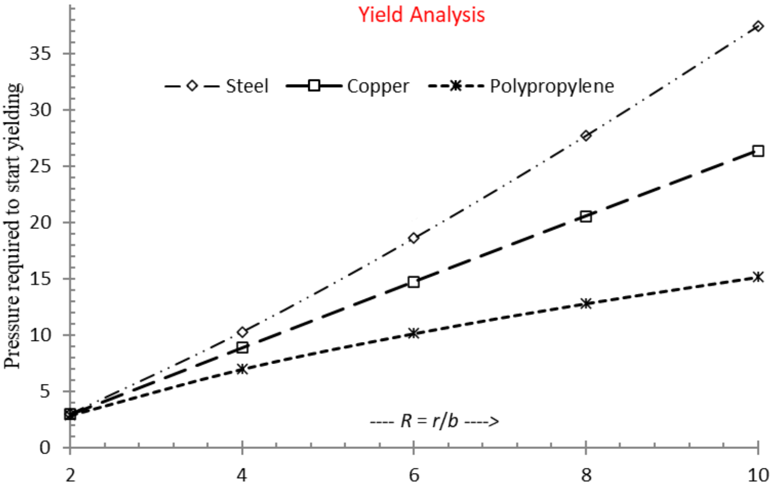

Figure 5. Pressure required to start yielding in spherical shell along radii ratio $R = r/b$.

pylene yields at lower values of pressure and causes damage in the spherical shell. Therefore, it is concluded that the spherical shell made of steel or copper has a longer life and more sustainability.

CONCLUSIONS

While studying the elastic-plastic deformation of the spherical shell, we have reached the following results.

- The stress components for a spherical shell under internal pressure can be found along radii ratio $R = r/b$.
- The pressure required to start yielding in the spherical shell can be calculated for different type of materials.

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