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ELASTOPLASTIC STRESS DEFORMATION IN AN ANNULAR DISK MADE OF ISOTROPIC MATERIAL AND SUBJECTED TO UNIFORM PRESSURE

ELASTOPLASTIČNI NAPON I DEFORMACIJA KOD KRUŽNOG DISKA OD IZOTROPNOG MATERIJALA RAVNOMERNO OPTEREĆENOG NA PRITISAK

Originalni naučni rad / Original scientific paper
UDK /UDC:

Rad primljen / Paper received: 1.12.2021

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Keywords
- isotropic structure
- pressure
- stress concentrations
- thickness
- disk

Ključne reči
- izotropna struktura
- pritisak
- koncentracija napona
- debljina
- disk

Abstract

This article deals with the study of elastoplastic stress deformation in an annular disk made of isotropic material under uniform pressure. Seth's transition theory and generalised strain measure are used for finding the governing equation. Analytical solutions are presented for the annular disk made of compressible/incompressible material. The effects of different pertinent parameters (i.e. thickness, pressure) are considered. The behaviour of stress distribution, and pressure are investigated. From the obtained results, it is noticed that the annular disk of compressible material requires higher pressure at the inner surface as compared to the incompressible material. The value of hoop stress is maximal at the outer surface of the annular disc of compressible material as compared to incompressible material.

Izvod

U ovom radu se istražuje elastoplastičan napon i deformacija kod kružnog diska od izotropnog materijala pod dejstvom jednolikog pritiska. *Teorija prelaznih napona Seta i mera generalisanih deformacija se koriste za iznalaženje potrebnih izraza. Predstavljena su analitička rešenja za kružni disk od stišljivog/nestišljivog materijala. Razmatraju se uticaji raznih relevantnih parametara (na pr. debljine, pritiska). Istraženo je ponašanje raspodele napona i pritiska. Prema dobijenim rezultatima, primećuje se da kružni disk od stišljivog materijala zahteva veći pritisak na unutrašnjoj površini, u poređenju sa nestišljivim materijalom. Vrednost obimskog napona je maksimalna na spoljnjoj površini kružnog diska od stišljivog materijala, u poređenju sa nestišljivim materijalom.*

INTRODUCTION

The analysis of rotating disks is an important aspect to be considered because of their numerous practical applications in chemical processing, mechanical, and aerospace industries such as high speed gear engines, flywheels, compressors, turbo generators, sink fits, pumps, steam turbine, and computer disks, etc. There are also many other applications like ship generators, rotors, and automotive braking systems. The solution for thin isotropic discs can be found in most of standard elasticity and plasticity textbooks /1, 2, 3, 6, 7/. Parmaksigoğlu et al. /8/ analysed the problem of plastic stress distribution in a rotating disk with shaft, subjected to radial temperature gradient, under the assumptions of Tresca's yield condition, its associated flow rule, and linear strain hardening. Kasayapanand /9/ investigated exact stress, strain, and displacement components of triple-composite rotating disks from different composite materials by using theory of infinitesimal linear elasticity. You et al. /10/ have investigated stress distribution in circular disks made of functionally graded materials subjected to internal and external pressure. Lin /11/ analysed rotating functionally graded annular disk with exponentially-varying profile and properties. Seth's transition theory /4, 5/ includes classical macroscopic solving problems in elasticity, plasticity, creep and relaxation, and assumes semi-empirical yield conditions. In this paper, we investigate stress distribution in an annular disk made of isotropic material under uniform pressure by using transition theory. Results are discussed and depicted graphically.

MATHEMATICAL MODEL AND GOVERNING EQUATION

Let us consider an annular thin isotropic disk made of isotropic materials with inner radius a, outer radius b, subjected to internal pressure p and is assumed to be symmetric with respect to the mid plane, and the profile of isotropic annular disc is given in form:

$$h = h_i (r/a)^k , \tag{1}$$

where: k is the thickness parameter, as shown in Fig. 1, respectively. The disk is taken to be sufficiently small so that the disk is in a state of plane stress.

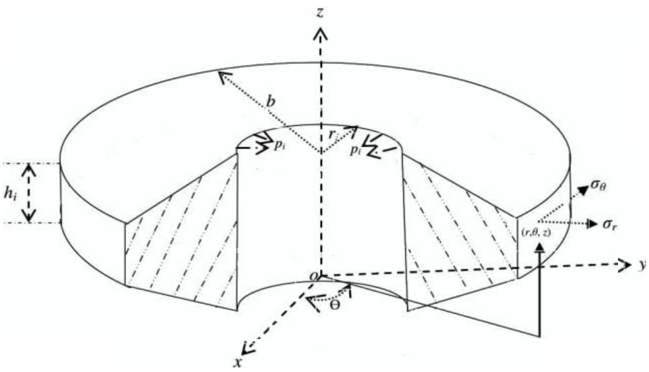

Figure 1. Geometrical configuration of annular disk.

Basic governing equation

Components of displacement (r,θ,z) are given by /4/:

$$u = r(1-\beta)\,,\quad v = 0\,,\quad w = dz\,. \tag{2}$$

The strain components for infinitesimal deformation are given by:

$$e_{rr}^A \equiv \frac{\partial u}{\partial r} = [1-(r\beta'+\beta)]\,,\quad e_{\theta\theta}^A \equiv \frac{u}{r} = (1-\beta)\,,$$

$$e_{zz}^A \equiv \frac{\partial w}{\partial z} = d\,,\quad e_{r\theta}^A = e_{\theta z}^A = e_{zr}^A = 0\,, \tag{3}$$

where: $\beta' = d\beta/dr$. Generalised strain measures are given by /4 , 5/:

$$e_{ii}^M = \int_0^{\varepsilon_{ii}^A} [1-2e_{ii}^A]^{\frac{n}{2}-1} de_{ii}^A = \frac{1}{n}\left[1-(1-2e_{ii}^A)^{n/2}\right],\ i = 1,2,3, \tag{4}$$

where: n = –2, –1, 0, 1, 2 give Green, Cauchy, Hencky, Swainger, and Almansi measures, respectively; and e_{ii}^A are Almansi finite strain components. Using Eq.(3) in Eq.(4), the generalised components of strain are obtained as:

$$e_{rr} = \frac{1}{n}\left[1-[2(r\beta'+\beta)-1]^{n/2}\right],\quad e_{\theta\theta} = \frac{1}{n}\left[1-(2\beta-1)^{n/2}\right],$$

$$e_{zz} = \frac{1}{n}\left[1-(1-2d)^{n/2}\right],\quad e_{r\theta} = e_{\theta z} = e_{zr} = 0\,, \tag{5}$$

where: n is the measure. The stress-strain relations for isotropic material are given by /1, 6/:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij}\,,\quad (i,j = 1,2,3)\,. \tag{6}$$

Equation (6) becomes:

$$T_{rr} = \frac{2\lambda\mu}{\lambda+2\mu}[e_{rr}+e_{\theta\theta}] + 2\mu e_{rr}\,,\quad T_{\theta\theta} = \frac{2\lambda\mu}{\lambda+2\mu}[e_{rr}+e_{\theta\theta}] + 2\mu e_{\theta\theta}\,,$$

$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0\,. \tag{7}$$

Substituting Eq.(5) into Eq.(7), we get

$$T_{rr} = \frac{2\mu}{n}\left[3-2c-(2-c)\left\{2\beta(P+1)-1\right\}^{n/2} - (1-c)(2\beta-1)^{n/2}\right],$$

$$T_{\theta\theta} = \frac{2\mu}{n}\left[3-2c-(1-c)\left\{2\beta(P+1)-1\right\}^{n/2} - (2-c)(2\beta-1)^{n/2}\right],$$

$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0\,, \tag{8}$$

where: $r\beta' = \beta P$ (P is function of β, and β is function of r); and $c = 2\mu/(\lambda + 2\mu)$. For an isotropic disk, centrifugal forces are zero, therefore, equations of equilibrium become:

$$\frac{d}{dr}(rhT_{rr}) - hT_{\theta\theta} = 0\,, \tag{9}$$

where: h is the thickness of isotropic hyperbolic disk. Using Eq.(8) and Eq.(9), we get a nonlinear differential equation in β as:

$$(2-c)n\beta^2\{2\beta(P+1)-1\}^{\frac{n}{2}-1}\frac{dP}{d\beta} = (2\beta-1)^{n/2} - \{2\beta(P+1)-1\}^{n/2} -$$

$$-(1-c)nP\beta(2\beta-1)^{\frac{n}{2}-1} + \frac{rh'}{h}\Big\{3-2c-\{2\beta(P+1)-1\}^{n/2} -$$

$$-(2\beta-1)^{n/2}(1-c)\Big\}\,, \tag{10}$$

where: $r\beta' = \beta P$; and $\beta' = d\beta/dr$.

Boundary conditions

The boundary conditions are taken as:

$$T_{rr} = -p \quad \text{at} \quad r = a,$$

and

$$T_{rr} = 0 \quad \text{at} \quad r = b, \tag{11}$$

where: T_{rr} and p are the radial stress and pressure applied at the internal surface of the annular disk.

SOLUTIONS

For finding the plastic stress distribution, the transition function Ψ is taken through the principal stresses /4, 5, 12-26/ at the transition point $P \to \pm\infty$ as:

$$\Psi = \frac{n}{2\mu}T_{\theta\theta} = 3-2c-(1-c)\{2\beta(P+1)-1\}^{n/2} -$$

$$-(2-c)(2\beta-1)^{n/2}\,. \tag{12}$$

Taking the logarithmic differentiation of Eq.(12) with respect to r, we get

$$\frac{d(\log\Psi)}{dr} = \frac{(2-c)n\beta P(2\beta-1)^{\frac{n}{2}-1} - (1-c)n\{2\beta(P+1)-1\}^{\frac{n}{2}-1}\left\{\beta P(P+1) + \beta^2 P\dfrac{dP}{d\beta}\right\}}{r\left\{3-2c-\{2\beta(P+1)-1\}^{n/2}(1-c) - (2\beta-1)^{n/2}(2-c)\right\}}\,. \tag{13}$$

Putting Eq.(10) into Eq.(13) and taking $P \to \pm\infty$ and then integrating, we get:

$$\Psi = \left(\frac{A}{h}\right) r^{\nu-1}\,, \tag{14}$$

where: $\nu = (1 - c)/(2 - c)$. Therefore, Eq.(12) and Eq.(14) become:

$$T_{\theta\theta} = \left(\frac{2\mu}{n}\right)\frac{Ar^{\nu-1}}{h}\,. \tag{15}$$

Substituting Eq.(15) into Eq.(9) and using Eq.(1), after that integrating, we get:

$$T_{rr} = \left(\frac{2\mu}{n\nu}\right)\frac{Aa^k r^{\nu-k-1}}{h_i} + \frac{Ba^k}{h_i r^{k+1}}\,. \tag{16}$$

By applying boundary conditions Eq.(11) into Eq.(16), we get:

$$A = \left(\frac{n}{2\mu}\right)\frac{ph_0 a\nu}{(b^\nu - a^\nu)} \quad \text{and} \quad B = -\frac{pab^\nu h_0}{(b^\nu - a^\nu)}\,.$$

Substituting the values of A and B into Eq.(15) and Eq.(16), we get the stress at the transitional stage:

$$T_{rr}=\frac{pR_0^{k+1}(R^{\nu}-1)}{R^{k+1}(1-R_0^{\nu})}\,,\quad T_{\theta\theta}=\frac{p\nu R_0^{k+1}R^{\nu-k-1}}{(1-R_0^{\nu})}\,,\tag{17}$$

and
$$T_{\theta\theta}-T_{rr}=\frac{pR_0^{k+1}}{R^{(k+1)}(1-R_0^{\nu})}[(\nu-1)R^{\nu}+1]\,,\tag{18}$$

where: $R = r/b$, and $R_0 = a/b$.

Initial yielding stage: to find the maximal value of $|T_{\theta\theta}-T_{rr}|$, it is seen from Eq.(18) that the first derivative of Eq. (18) with respect to R is:

$$\frac{d}{dR}(T_{\theta\theta}-T_{rr})=\frac{pR_0^{k+1}}{(1-R_0^{\nu})}\left[(\nu-1)(\nu-k-1)R^{\nu-k-2}-(k+1)R^{-k-2}\right]$$

which is zero at $R=\left[\frac{k+1}{(\nu-1)(\nu-k-1)}\right]^{1/\nu}\cong R_i$.

Therefore, the second derivative with respect to R is given by:

$$\frac{d^2}{dR^2}(T_{\theta\theta}-T_{rr})=\frac{pR_0^{k+1}}{(1-R_0^{\nu})}\Big[(\nu-1)(\nu-k-1)(\nu-k-2)R^{\nu-k-3}+ +(k+1)(k+2)R^{-k-3}\Big]\,,$$

and it is negative $R = R_i$ for $k < -1$. Consequently, $T_{\theta\theta} - T_{rr}$ is maximum at $R = R_i$. Hence, yielding in the convergent disk will take place at $R = R_i$ depending on the values of $\nu\in$ (0, 0.5) and. $k=\frac{R_i^{\nu}(\nu-1)^2-1}{1+R_i^{\nu}(\nu-1)}$. For example, if we take ν = 0.27, 0.4999, then k = –1.41, –1.33, respectively. Furthermore, the validation value of k is applicable if $k \le -1$ at the internal surface, $R = R_i$. Eq.(19) becomes:

$$|T_{\theta\theta}-T_{rr}|_{R\cong R_i}=\left|\frac{pR_0^{k+1}}{(1-R_0^{\nu})}\left(\frac{\nu}{\nu-k-1}\right)\left[\frac{k+1}{(\nu-1)(\nu-k-1)}\right]^{-\frac{(k+1)}{\nu}}\right|\cong$$
$$\cong Y \text{ (yielding stress)}\,,$$

and the required pressure for yielding is given by:

$$p_i=\frac{p}{Y}=\left|\frac{(1-R_0^{\nu})}{R_0^{k+1}}\left(\frac{\nu-k-1}{\nu}\right)\left[\frac{(\nu-1)(\nu-k-1)}{k+1}\right]^{-\frac{(k+1)}{\nu}}\right|.$$

Therefore, Eq.(17) becomes:

$$\sigma_r=\frac{p_iR_0^{k+1}(R^{\nu}-1)}{R^{k+1}(1-R_0^{\nu})}\,,\quad \sigma_\theta=\frac{p_i\nu R_0^{k+1}R^{\nu-k-1}}{(1-R_0^{\nu})}\,,\tag{19}$$

where: $\sigma_r = T_{rr}/Y$; $\sigma_\theta = T_{\theta\theta}/Y$.

The convergent disc becomes fully plastic ($\nu \to 1/2$) at the external surface (i.e. R = 1) for

$$|T_{\theta\theta}-T_{rr}|_{R=1}=\left|\frac{pR_0^{k+1}}{2(1-\sqrt{R_0})}\right|\cong Y^* \text{ (yielding stress).}$$

The pressure for fully-plastic stage:

$$p_f=\frac{p}{Y^*}=\frac{2(1-\sqrt{R_0})}{R_0^{k+1}}\,.\tag{20}$$

Eq.(19) for the fully-plastic stage becomes:

$$\sigma_r=\frac{p_fR_0^{k+1}(\sqrt{R}-1)}{R^{k+1}(1-\sqrt{R_0})}\,,\quad \sigma_\theta=\frac{p_fR_0^{k+1}}{2R^{k+\frac{1}{2}}(1-\sqrt{R_0})}\,.\tag{21}$$

NUMERICAL ILLUSTRATION AND DISCUSSION

To investigate the effect of stress distribution and pressure for the annular disc made of isotropic materials (say, rubber ν = 0.5; steel ν = 0.27) from /1/ the following numerical values are taken: a = 0.5 (inner radius); b =1 (outer radius); $r \in (a,b)$ and k < 0. Table 1 shows at $R = R_i$, whenever Poisson's ratios ν = 0.4999 (rubber), 0.27 (steel), then k = –1.33 and –1.414 yielding occurs at the internal surface R_i = 0.5. It has been observed that annular disc made of incompressible material (say steel) requires high percentage values of pressure (i.e., P = 12.3 %) to become fully plastic as compared to the annular disk of compressible material (say rubber).

In Fig. 2, the curve is drawn between pressure required for the initial yielding stage/fully-plastic stage at the internal surface of the annular disk along the radio ratio $R_0 = a/b$. It has been observed that the annular disk of compressible material requires a higher pressure at the inner surface as compared to the incompressible material for initial yielding/ fully-plastic stage. Figure 3 demonstrates the behaviour of the stress distribution vs. radii ratio $R = r/b$. It has been observed that hoop stress is maximal at the outer surface of the annular disc of compressible material as compared to incompressible material. Furthermore, the annular disc of compressible material (i.e., rubber) is more comfortable than that of the steel material.

ACKNOWLEDGEMENT

This research paper is especially dedicated to the Society for Structural Integrity and Life and the IMS Institute for Materials Testing, Serbia.

Table 1. Percentage in pressure required for initial/fully plastic stage.

Materials	Yielding starts at the bore, $R = R_i$			Percentage increase in pressure, P (%)
	thickness k	pressure p_i	pressure p_f	$P=\left(\frac{p_f-p_i}{p_i}\right)\cdot 100$
rubber, ν = 0.4999 (compressible)	–1.33	0.453	0.48	7.2 %
steel, ν = 0.27 (incompressible)	–1.41	0.432	0.48	12.3 %

where: $P=\left(\frac{p_f-p_i}{p_i}\right)\cdot 100$ is the percentage (%) increase in pressure from initial yielding stage to become fully plastic stage.

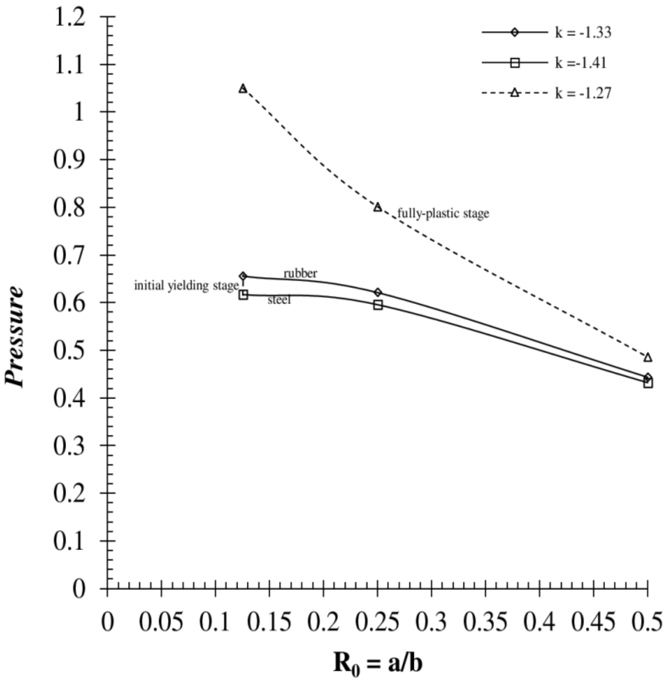

Figure 2. Graphical results of pressure vs. radii ratio $R_0 = a/b$.

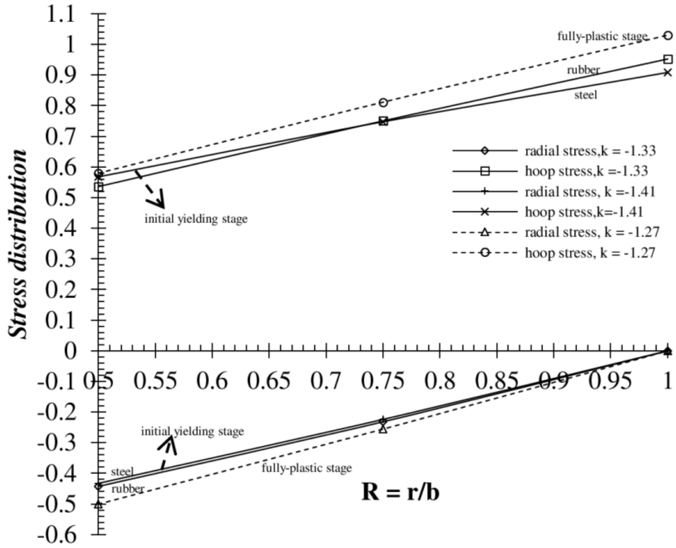

Figure 3. Graphical result of stress distribution vs. radii ratio $R = r/b$.

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