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EFFECT OF HYDROSTATIC INITIAL STRESS IN A FUNCTIONALLY GRADED HYGRO-THERMOELASTIC HALF SPACE

UTICAJ HIDROSTATIČKOG INICIJALNOG NAPONA U HIGROTERMOELASTIČNOM POLUPROSTORU FUNKCIONALNOG KOMPOZITNOG MATERIJALA

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- functionally graded material
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Ključne reči
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- funkcionalni kompozitni materijal
- normalni mod (režim)
- koncentracija vlage

Abstract

The current investigation deals with the study of surface wave propagation in hygrothermoelastic medium under hydrostatic initial stress. The wave equations in terms of displacement, temperature and moisture concentration are solved analytically. The non-homogeneous mechanical and thermal properties of functionally graded material are supposed to be in x direction. The components of displacement, mechanical stress, moisture concentration and temperature distribution in the medium are evaluated using normal mode analysis technique. The deformation which is caused due to mechanical force along the free surface of hygrothermoelastic solid with hydrostatic initial stress in medium is also discussed. The analytical results obtained have also been depicted graphically to show the effect of non-homogeneous parameters and hydrostatic initial stress parameters.

Izvod

Ova istraživanja se bave proučavanjem prostiranja površinskog talasa u higrotermoelastičnoj sredini pod dejstvom inicijalnog hodrostatičkog napona. Jednačine talasa u smislu pomeranja, temperature i koncentracije vlage se rešavaju analitički. Nehomogene mehaničke i termičke osobine funkcionalnog kompozitnog materijala se pretpostavljaju u pravcu x ose. Komponente pomeranja, mehaničkog napona, koncentracije vlage i raspodele temperature date sredine se određuju primenom postupka analize normalnog moda. Takođe je data diskusija o deformaciji koju izaziva mehanička sila duž slobodne površine higrotermoelastičnog čvrstog tela sa hidrostatičkim inicijalnim naponom u datoj sredini. Dobijeni analitički rezultati su takođe predstavljeni grafički, gde se vidi uticaj nehomogenih parametara i parametara hidrostatičkog inicijalnog napona.

INTRODUCTION

Any mechanical stress applied to the solid material significantly affects the moisture as well as temperature distribution of the solid. It is fundamentally observed that there is strong correlation of change in moisture and temperature to the mechanical deformation of a solid indicating the importance of present study on hygrothermoelasticity. The concept of hygrothermoelasticity emerged when the solids are under the influence of moisture and heat effects.

Significant contribution in the field of hygrothermoelasticity was given by Sih et al. /1/. Sih et al. /2/ analytically investigated composite exhibiting transient stress keeping thermal diffusion coefficients constant and moisture diffusion coefficient dependent on temperature. Cross coupling between two parameters i.e., heat and moisture affecting composites was studied further in detail by Szekeres /3, 4/. All the subcases of coupled, convection with diffusion of cross coupled moisture and heat in hygroscopic composite materials were finely formulated by Szekeres and Engelbrecht /5/. The thermochemical and mechanical damage of concrete was discussed using modified isotropic non-local damage theory. Macroscopic balance of linear momentum and model governing equation were also presented by Gawin et al. /6/. The effect of active stiffening in correlation with piezo-electricity, hygroscopic, and thermoelastic material was extensively discussed by Raja et al. /7/. To study the effect of multiphase composites of a coupled micro- macro mechanical analysis, Aboudi and Williams /8/ developed this theory. Rao and Sinha /9/ carried out research for multidirectional composites by using three dimensional analysis to check the effect of temperature and moisture on free vibrations. Huang et al. /10/ discussed nonlinear vibration and dynamic response of shear deformable laminated plates in hygrothermal environments. By modifying Tenchev formulation significance of free water flux in determining effect of fluid transport behaviour on concrete was extensively studied by Davie et al. /11/. Aoki et al. /12/ observed the combined effect of water absorption and thermal environment on compression. Wang and Pan /13/ carried out research on three dimensional quasi-steady-state temperature and moisture concentration of a constantly moving heat source and diffusion source which is an extension of the well known Jaegar-Rosenthal solution. Accurate three-dimensional model describing variation of vapour content and pressure with time and space was well developed by Davie et al. /14/. Chiba and Sugano /15/ obtained the analytical solution for the transient one way coupled temperature and moisture fields. Hosseini et al. /16/ presented the meshless local Petro-Galerkin (MLPG) method to discuss the two-dimensional coupled non Fick diffusion elasticity analysis. Mashat and Zenkour /17/ discussed the hygrothermal bending analysis of sector shaped annular plates with variable thickness. A verfied valid model was developed using finite element discretization in space for hygrothermal mechanical analysis of spalling by Benes and Stefan, /18/. Koniorczyk et al. /19/ discussed the modelling evolution of frost damage in fully saturated porous materials under hygrothermal conditions. Ahmed et al. /20/ investigated the behaviour of unidirectional CFRP composite plates under different moisture content conditions. Zhang and Li /21/ proposed the time fractional hygro-thermoelasticity theory in which coupled heat and moisture satisfy diffusion wave equation with time fractional derivation. Significant transient behaviour of hygrothermoelastic with reference to moisture and heat flux was observed by Zhang et al. /22/ using time fractional calculus theory. To obtain the steady state general solution for three dimensional hygrothermoelastic media, Zhao et al. /23/ used potential theory method. Tong et al. /24/ developed the steady state general solution for the two-dimensional hygrothermoelastic media. The plane wave propagation in hygrothermoelastic medium with hydrostatic initial stress was discussed by Ailawalia et al. /25/.

Gravitational forces, creeping and temperature gradient may be considered as the factors behind initial stress. Considerable significant research work is done in the field of wave propagation in unbounded medium solid by Chattopadhyay et al. /26/, Sidhu and Singh /27/, and Dey et al. /28/. Using Biot's linearization, Montanaro /29/ applied and proved theory given by Chadwick and Powdrill in a medium where hydrostatic initial stress is applied. Many authors: Singh et al. /30/, Othman and Song /31/, Singh /32/ have used the hydrostatic initial stress formulation to study the plane waves under generalized thermoelasticity. Mechanical force in combination with hydrostatic initial stress deformed the body to a large extent as observed by Ailawalia et al. /33/. Effect of irregular depth and hydrostatic initial stress on half space which is magneto-elastic as well as monoclinic on normal and shear stress, was summarized by Mistri et. al /34/. The governing equations of two temperature generalized thermoelasticity under hydrostatic initial stress was solved by Kumar and Singh, /35/.

Based on powder metallurgy, a potential structure regulating temperature variation was developed in Japan 1984, called as Functionally graded material (FGM) /36-38/. FGM's comprise of mixtures of materials based on their thermal conductivity which makes FGM a significant potential thermal barrier. Reddy and Chin /39/ discussed the thermomechanical analysis of functionally graded cylinders and plates. Sankar and Tzeng /40/ discussed the thermal stresses in functionally graded beams. Abbas et al. /41/ studied the LS model on electromagneto-thermoelastic response of an infinite functionally graded cylinder. Ghunghas et al. /42/ studied the influence of rotation and magnetic fields on a functionally graded thermoelastic solid. Kalkal et al. /43/ discussed the two-dimensional magneto-thermoelastic interactions in a micropolar functionally graded solid.

In the present research investigation, the authors studied the mechanical deformation in a non-homogeneous hygrothermoelastic medium under hydrostatic initial stress. The analytical expressions of displacement, stress, moisture concentration and temperature field are derived using normal mode technique. These results are also represented graphically to show the effect of non-homogeneous parameter and initial stress parameter on the components.

BASIC EQUATIONS

Following Hosseini et al. /16/ and Montanaro /29/, the constitutive relations, field equations, heat conduction and moisture diffusion for isotropic hygrothermoelastic solid with hydrostatic initial and in the absence of incremental body forces and heat sources are,

$$\sigma_{ji,j} = \rho \ddot{u}_i , \tag{1}$$

$$D_T T_{,ii} + D_T^m m_{,ii} - \dot{T} - \frac{\beta_{ij}^T T_0}{\rho c} \dot{u}_{j,j} = 0 , \tag{2}$$

$$D_m m_{,ii} + D_m^T T_{,ii} - \dot{m} - \frac{\beta_{ij}^m m_0 D_m}{k_m} \dot{u}_{j,j} = 0 , \tag{3}$$

where,

$$\beta_{ij}^T = \beta_T \delta_{ij} , \quad \beta_T = (3\lambda + 2\mu)\alpha_T , \tag{4}$$

$$\beta_{ij}^m = \beta_m \delta_{ij} , \quad \beta_m = (3\lambda + 2\mu)\alpha_m , \tag{5}$$

$$\sigma_{ij} = -P(\delta_{ij} + \varpi_{ij}) + C_{ijkl}\varepsilon_{kl} - \beta_{ij}^m m - \beta_{ij}^T T , \tag{6}$$

$$C_{ijkl} = \frac{2G\nu}{1-2\nu}\delta_{ij}\delta_{kl} + G\delta_{ik}\delta_{jl} + G\delta_{il}\delta_{jk} , \tag{7}$$

$$\varepsilon_{ij} = \frac{u_{j,i} + u_{i,j}}{2} , \quad \varpi_{ij} = \frac{u_{j,i} - u_{i,j}}{2} . \tag{8}$$

Here, σ_{ij}, ε_{ij}, u_i are components of stress, strain, and displacement, respectively; P is initial pressure; ρ is density; D_T temperature diffusivity; T temperature; m moisture concentration; T_0 reference temperature; c heat capacity; D_m is diffusion coefficient of moisture; D_T^m, D_m^T are coupled diffusivities; m_0 reference moisture; k_m moisture diffusivity; β_{ij}^T and β_{ij}^m are material coefficients arising on the account of coupling between stresses and temperature, respectively; α_T is coefficient of linear thermal expansion; α_m is coefficient of moisture expansion; and λ and μ are Lame's constants.

For a functionally graded material, the parameters λ, μ, P, ρ, β_m, β_T, and k_m are no longer constant, but become space dependent. Hence, we replace λ, μ, P, ρ, β_m, β_T, and k_m by $\lambda_0 f(\vec{x})$, $\mu_0 f(\vec{x})$, $P_0 f(\vec{x})$, $\rho_0 f(\vec{x})$, $\beta_{m0} f(\vec{x})$, $\beta_{T0} f(\vec{x})$, and $k_{m0} f(\vec{x})$, respectively, λ_0, μ_0, P_0, ρ_0, β_{m0}, β_{T0}, and k_{m0} are assumed to be constant; and $f(\vec{x})$ is a given dimensionless function of the space variable $\vec{x} = (x, y, z)$. Using these values of parameters in Eqs.(1)-(3) and the constitutive relation Eq.(6) takes the following form:

$$\sigma_{ji,j} = \rho_0 f(\vec{x})\ddot{u}_i , \tag{9}$$

$$D_T (f(\vec{x})T_{,ii}) + D_T^m (f(\vec{x})m_{,ii}) - \dot{T} f(\vec{x}) - f(\vec{x}) \frac{\beta_{ij}^T T_0}{\rho_0 c} \dot{u}_{j,j} = 0 , \tag{10}$$

$$D_m (f(\vec{x})m_{,ii}) + D_m^T (f(\vec{x})T_{,ii}) - f(\vec{x})\dot{m} - f(\vec{x}) \frac{\beta_{ij}^m m_0 D_m}{k_{m0}} \dot{u}_{j,j} = 0 , \tag{11}$$

$$\sigma_{ij} = f(\vec{x})[-P_0(\delta_{ij} + \varpi_{ij}) + C_{ijkl}\varepsilon_{kl} - \beta_{ij}^m m - \beta_{ij}^T T]. \qquad (12)$$

Here, the superposed dot denotes the derivative with respect to time, and comma denotes the derivative with respect to the space variable.

PROBLEM FORMULATION

We consider a rectangular coordinate system (x,y,z) with x-axis pointing vertically downward. The current study is restricted to the x-z plane. Thus all the field quantities are independent of the space variable y. So, the displacement components are taken as: $\vec{u} = (u,0,w)$, where $u = u(x,z,t)$, $w = w(x,z,t)$. It further reduces the equations of motion and coupled generalised equations of heat conduction and moisture diffusion Eqs.(9)-(11) and constitutive relation Eq.(12) in two dimensions and in the absence of body forces. It is also supposed that the material properties vary only in x direction. Hence, we take $f(\vec{x})$ as $f(x)$.

$$f(x)\left[(\lambda_0+2\mu_0)\frac{\partial^2 u}{\partial x^2}+\left(\lambda_0+\mu_0+\frac{P_0}{2}\right)\frac{\partial^2 w}{\partial x\partial z}+\left(\mu_0-\frac{P_0}{2}\right)\frac{\partial^2 u}{\partial z^2}-\beta_{m0}\frac{\partial m}{\partial x}-\beta_{T0}\frac{\partial T}{\partial x}\right]+$$
$$+\frac{\partial}{\partial x}f(x)\left[-P_0+\lambda_0\frac{\partial w}{\partial z}+(\lambda_0+2\mu_0)\frac{\partial u}{\partial x}-\beta_{m0}m-\beta_{T0}T\right]=\rho_0 f(x)\frac{\partial^2 u}{\partial t^2}, \qquad (13)$$

$$f(x)\left[(\lambda_0+2\mu_0)\frac{\partial^2 w}{\partial z^2}+\left(\lambda_0+\mu_0+\frac{P_0}{2}\right)\frac{\partial^2 u}{\partial x\partial z}+\left(\mu_0-\frac{P_0}{2}\right)\frac{\partial^2 w}{\partial x^2}-\beta_{m0}\frac{\partial m}{\partial z}-\beta_{T0}\frac{\partial T}{\partial z}\right]+$$
$$+\frac{\partial}{\partial x}f(x)\left[-P_0+\lambda_0\frac{\partial w}{\partial z}+(\lambda_0+2\mu_0)\frac{\partial u}{\partial x}-\beta_{m0}m-\beta_{T0}T\right]=\rho_0 f(x)\frac{\partial^2 u}{\partial t^2}, \qquad (14)$$

$$D_T\left[f(x)\nabla^2 T+\frac{\partial}{\partial x}f(x)\frac{\partial T}{\partial x}\right]+D_T^m\left[f(x)\nabla^2 m+\frac{\partial}{\partial x}f(x)\frac{\partial m}{\partial x}\right]-f(x)\frac{\partial T}{\partial t}-\frac{\beta_{T0}T_0}{\rho_0 c}f(x)\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]=0, \qquad (15)$$

$$D_m\left[f(x)\nabla^2 m+\frac{\partial}{\partial x}f(x)\frac{\partial m}{\partial x}\right]+D_m^T\left[f(x)\nabla^2 T+\frac{\partial}{\partial x}f(x)\frac{\partial T}{\partial x}\right]-f(x)\frac{\partial m}{\partial t}-\frac{\beta_{m0}m_0D_m}{k_{m0}}f(x)\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right]=0. \qquad (16)$$

The stresses arising from Eq.(6) can be expressed as

$$\sigma_{xx}=f(x)\left[-P_0+\lambda_0\frac{\partial w}{\partial z}+(\lambda_0+2\mu_0)\frac{\partial u}{\partial x}-\beta_{m0}m-\beta_{T0}T\right], \qquad (17)$$

$$\sigma_{xz}=f(x)\left[\left(\mu_0+\frac{P_0}{2}\right)\frac{\partial u}{\partial z}+\left(\mu_0-\frac{P_0}{2}\right)\frac{\partial w}{\partial x}\right], \qquad (18)$$

$$\sigma_{zx}=f(x)\left[\left(\mu_0-\frac{P_0}{2}\right)\frac{\partial u}{\partial z}+\left(\mu_0+\frac{P_0}{2}\right)\frac{\partial w}{\partial x}\right], \qquad (19)$$

$$\sigma_{zz}=f(x)\left[-P_0+\lambda_0\frac{\partial u}{\partial x}+(\lambda_0+2\mu_0)\frac{\partial w}{\partial z}-\beta_{m0}m-\beta_{T0}T\right], \qquad (20)$$

EXPONENTIAL VARIATION

Let us assume $f(x) = e^{-nx}$, where n is a non-dimensional parameter. Hence, the material properties vary exponentially along the x direction. The governing equations can be rewritten in the dimensionless form by introducing the following dimensionless parameters:

$$x'=\frac{x}{l},\ z'=\frac{z}{l},\ u'=\frac{u}{l},\ w'=\frac{w}{l},\ t'=\frac{D_m}{l^2}t,\ m'=m,$$
$$T'=\frac{T}{T_0},\ \sigma'_{ij}=\frac{\sigma_{ij}}{\lambda_0+2\mu_0},\ P'_{ij}=\frac{P_0}{\lambda_0+2\mu_0}, \qquad (21)$$

where: the quantity l has dimension of length.

Using Eq.(21) in Eqs.(13)-(16) and Eqs.(17)-(20), we get the following dimensionless equations (after dropping the primes),

$$\frac{\partial^2 u}{\partial x^2}+a_1\frac{\partial^2 w}{\partial x\partial z}+a_2\frac{\partial^2 u}{\partial z^2}-a_3\frac{\partial m}{\partial x}-a_4\frac{\partial T}{\partial x}+nP_0-na_5\frac{\partial w}{\partial z}-$$
$$-n\frac{\partial u}{\partial x}+na_3 m+na_4 T=a_6\frac{\partial^2 u}{\partial t^2}, \qquad (22)$$

$$\frac{\partial^2 w}{\partial z^2}+a_1\frac{\partial^2 u}{\partial x\partial z}+a_2\frac{\partial^2 w}{\partial x^2}-a_3\frac{\partial m}{\partial z}-a_4\frac{\partial T}{\partial z}-na_7\frac{\partial u}{\partial z}-$$
$$-na_2\frac{\partial w}{\partial x}=a_6\frac{\partial^2 w}{\partial t^2}, \qquad (23)$$

$$\left(\nabla^2 T-n\frac{\partial T}{\partial x}\right)+a_8\left(\nabla^2 m-n\frac{\partial m}{\partial x}\right)-a_9\frac{\partial T}{\partial t}-a_{10}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)=0, \qquad (24)$$

$$\left(\nabla^2 m-n\frac{\partial m}{\partial x}\right)+a_{11}\left(\nabla^2 T-n\frac{\partial T}{\partial x}\right)-\frac{\partial m}{\partial t}-a_{12}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)=0, \qquad (25)$$

$$\sigma_{xx}=e^{-nx}\left[-P_0+a_5\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}-a_3 m-a_4 T\right], \qquad (26)$$

$$\sigma_{zz}=e^{-nx}\left[-P_0+a_5\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}-a_3 m-a_4 T\right], \qquad (27)$$

$$\sigma_{xz}=e^{-nx}\left[a_7\frac{\partial u}{\partial z}+a_2\frac{\partial w}{\partial x}\right], \qquad (28)$$

where: $a_1=\dfrac{\lambda_0+\mu_0}{\lambda_0+2\mu_0}+\dfrac{P_0}{2}$; $a_2=\dfrac{\mu_0}{\lambda_0+2\mu_0}-\dfrac{P_0}{2}$; $a_3=\dfrac{\beta_{m0}}{\lambda_0+2\mu_0}$; $a_4=\dfrac{\beta_{T0}T_0}{\lambda_0+2\mu_0}$; $a_5=\dfrac{\lambda_0}{\lambda_0+2\mu_0}$; $a_6=\dfrac{\rho_0 D_m^2}{(\lambda_0+2\mu_0)l^2}$; $a_7=\dfrac{\mu_0}{\lambda_0+2\mu_0}+\dfrac{P_0}{2}$; $a_8=\dfrac{D_T^m}{D_T T_0}$; $a_9=\dfrac{D_m}{D_T}$; $a_{10}=\dfrac{\beta_{T0}D_m^2}{D_T\rho_0 c}$; $a_{11}=\dfrac{D_m^T T_0}{D_m}$; $a_{12}=\dfrac{\beta_m m_0 D_m}{k_{m0}}$.

Solution of the problem

In this part, we use normal mode analysis technique to find the solution of the considered physical variables in the following form:

$$(u,w,m,T,\sigma_{ij})(x,y,t)=(u^*,w^*,m^*,T^*,\sigma_{ij}^*)(x)e^{\omega t+\imath bz}, \qquad (29)$$

where: ω is the complex frequency; b is the wave number in z direction; and u^*, w^*, m^*, T^*, σ_{ij}^* are the amplitudes of the field quantities.

Using Eq.(29) in Eqs.(22)-(28), we obtain the following equations,

$$(D^2-nD-d_1)u^*+\imath b(a_1D-na_5)w^*-(D-n)a_3m^*-\\-(D-n)a_4T^*=-nP_0\,, \tag{30}$$

$$\imath b(a_1D-na_7)u^*+(a_2D^2-na_2D-d_2)w^*-a_3\imath bm^*-\\-a_4\imath bT^*=0\,, \tag{31}$$

$$d_3Du^*+d_3\imath bw^*-a_8(D^2-nD-b^2)m^*-\\-(D^2-nD-d_4)T^*=0\,, \tag{32}$$

$$d_5Du^*+d_5\imath bw^*-(D^2-nD-d_6)m^*-\\-a_{11}(D^2-nD-b^2)T^*=0\,, \tag{33}$$

$$\sigma_{xx}^*=e^{-nx}\left[\frac{-P_0}{e^{\omega t+\imath bz}}+a_5\imath bw^*+Du^*-a_3m^*-a_4T^*\right], \tag{34}$$

$$\sigma_{zz}^*=e^{-nx}\left[\frac{-P_0}{e^{\omega t+\imath bz}}+a_5Du^*+\imath bw^*-a_3m^*-a_4T^*\right], \tag{35}$$

$$\sigma_{xz}^*=e^{-nx}\left[a_2Dw^*+a_7\imath bu^*\right], \tag{36}$$

where: $d_1 = a_2b^2 + a_6\omega^2$, $d_2 = b^2 + a_6\omega^2$, $d_3 = a_{10}\omega$, $d_4 = b^2 + a_9\omega^2$, $d_5 = a_{12}\omega$, $d_6 = b^2 + \omega$.

On solving the Eqs.(30)-(33), we get the following eighth degree equation,

$$[D^8+nf_1D^7+f_2D^6+nf_3D^5+f_4D^4+nf_5D^3+f_6D^2+\\+nf_7D+f_8](u^*,w^*,m^*,T^*)=-nP_0t_{36}\,, \tag{37}$$

where: f_i (i = 1 … 8) are listed in Appendix *A*. The solution of Eq.(37) which is bounded as $x \to \infty$, is given by

$$u^*(x)=\textstyle\sum_{i=1}^4 A_i(a,\omega)e^{-k_ix}+\xi_1\,, \tag{38}$$

$$w^*(x)=\textstyle\sum_{i=1}^4 \imath bH_{1i}A_i(a,\omega)e^{-k_ix}+\xi_2\,, \tag{39}$$

$$m^*(x)=\textstyle\sum_{i=1}^4 H_{2i}A_i(a,\omega)e^{-k_ix}+\xi_3\,, \tag{40}$$

$$T^*(x)=\textstyle\sum_{i=1}^4 H_{3i}A_i(a,\omega)e^{-k_ix}+\xi_4\,, \tag{41}$$

where: k_i (i = 1,2,3,4) are the roots of Eq.(37); and A_i (a,ω) (i = 1,2,3,4) are the parameters, depending on a and ω, and

$$H_{1i}=\frac{t_{24}k_i^5-nt_{25}k_i^4+t_{26}k_i^3-nt_{27}k_i^2+t_{28}k_i-nt_{29}}{t_{30}k_i^6-nt_{31}k_i^5+t_{32}k_i^4-nt_{33}k_i^3+t_{34}k_i^2-nt_{35}k_i+t_{36}}\,, \tag{42}$$

$$H_{2i}=\frac{(a_1k_i^3+nt_6k_i^2+t_7k_i-nt_8)-(a_2k_i^4+2na_2k_i^3+t_9k_i^2-nt_{10}k_i+t_{11})H_{1i}}{t_{12}k_i^2+nt_{12}k_i+t_{13}} \tag{43}$$

$$H_{3i}=\frac{-d_5k_i-d_5b^2H_{1i}-(k_i^2+nk_i-d_6)H_{2i}}{a_{11}(k_i^2+nk_i-b^2)}\,, \tag{44}$$

$$\xi_1=\frac{-nP_0t_{36}}{f_8}\,, \tag{45}$$

$$\xi_2=J_1\xi_1\,,\quad J_1=\frac{-\imath bnt_{29}}{t_{36}}\,, \tag{46}$$

$$\xi_3=J_2\xi_1\,,\quad J_2=\frac{nt_{29}t_{11}}{t_{36}t_{13}}-\frac{nt_8}{t_{13}}\,, \tag{47}$$

$$\xi_4=J_3\xi_1\,,\quad J_3=\frac{-d_5nt_{29}}{t_{36}a_{11}}-\frac{nd_6}{a_{11}b^2}\left(\frac{t_{29}t_{11}}{t_{36}t_{13}}-\frac{t_8}{t_{13}}\right). \tag{48}$$

Normal mode analysis of the stress components yields the following:

$$\sigma_{xx}^*=e^{-nx}[\phi(t,z)-\textstyle\sum_{i=1}^4 A_i(a,\omega)e^{-k_ix}U_i+\xi_5]\,, \tag{49}$$

$$\sigma_{zz}^*=e^{-nx}[\phi(t,z)-\textstyle\sum_{i=1}^4 A_i(a,\omega)e^{-k_ix}V_i+\xi_6]\,, \tag{50}$$

$$\sigma_{xz}^*=e^{-nx}[\textstyle\sum_{i=1}^4 A_i(a,\omega)e^{-k_ix}W_i\imath b+a_7\imath b\xi_1]\,, \tag{51}$$

where,
$$U_i=[a_5b^2H_{1i}+k_i+a_3H_{2i}+a_4H_{3i}]\,, \tag{52}$$

$$V_i=[a_5k_i+b^2H_{1i}+a_3H_{2i}+a_4H_{3i}]\,, \tag{53}$$

$$W_i=[a_2-a_7k_iH_{1i}]\,, \tag{54}$$

$$\xi_5=(a_5\imath bJ_1-a_3J_2-a_4J_3)\xi_1\,, \tag{55}$$

$$\xi_6=(\imath bJ_1-a_3J_2-a_4J_3)\xi_1\,, \tag{56}$$

where: $\phi(t,z)=\dfrac{-P_0}{e^{\omega t+\imath bz}}$.

Boundary conditions

To determine the constants A_i (i = 1,2,3,4), the boundary conditions at the free surface x = 0 are given by:

(i) $$\sigma_{xx}=-Fe^{\omega t+\imath bz}\,, \tag{57}$$

(ii) $$\sigma_{xz}=0\,, \tag{58}$$

(iii) $$\frac{\partial T}{\partial x}=0\,, \tag{59}$$

(iv) $$\frac{\partial m}{\partial x}=0\,, \tag{60}$$

where: F is the magnitude of the constant force applied to the boundary. Using Eqs.(38)-(41) and Eqs.(49)-(51) in the boundary conditions Eqs.(57)-(60), we get four equations of four unknowns as:

$$U_1A_1+U_2A_2+U_3A_3+U_4A_4=\phi(t,z)+\xi_5+F\,, \tag{61}$$

$$W_1A_1+W_2A_2+W_3A_3+W_4A_4=-a_7\xi_1\,, \tag{62}$$

$$R_1A_1+R_2A_2+R_3A_3+R_4A_4=0\,, \tag{63}$$

$$S_1A_1+S_2A_2+S_3A_3+S_4A_4=0\,, \tag{64}$$

where: $R_i=H_{3i}k_i$; $S_i=H_{2i}k_i$.

Solving Eqs.(61)-(64) which can be written in the matrix form as:

$$\begin{bmatrix}U_1&U_2&U_3&U_4\\W_1&W_2&W_3&W_4\\R_1&R_2&R_3&R_4\\S_1&S_2&S_3&S_4\end{bmatrix}\begin{bmatrix}A_1\\A_2\\A_3\\A_4\end{bmatrix}=\begin{bmatrix}\phi(t,z)+F+\xi_5\\-a_7\xi_1\\0\\0\end{bmatrix}. \tag{65}$$

Solution of the system Eq.(65) provides us the values of A_i (i = 1,2,3,4) as follows:

$$A_i=\frac{\Delta_i}{\Delta}\,,\ (i = 1,2,3,4), \tag{66}$$

where,

$$\Delta=U_1[W_2(R_3S_4-S_3R_4)-W_3(R_2S_4-S_2R_4)+W_4(R_2S_3-S_2R_3)]-\\-U_2[W_1(R_3S_4-S_3R_4)-W_3(R_1S_4-S_1R_4)+W_4(R_1S_3-S_1R_3)]+\\+U_3[W_1(R_2S_4-S_2R_4)-W_2(R_1S_4-S_1R_4)+W_4(R_1S_2-S_1R_2)]-\\-U_4[W_1(R_2S_3-S_2R_3)-W_2(R_1S_3-S_1R_3)+W_3(R_1S_2-S_1R_2)] \tag{67}$$

$$\Delta_1=(\phi(t,z)+F+\xi_5)[W_2(R_3S_4-S_3R_4)-W_3(R_2S_4-S_2R_4)+\\+W_4(R_2S_3-S_2R_3)]+a_7\xi_1[U_2(R_3S_4-S_3R_4)-U_3(R_2S_4-S_2R_4)+$$

$$+U_4(R_2S_3-S_2R_3)]\,, \tag{68}$$

$$\Delta_2=(\phi(t,z)+F+\xi_5)[W_1(R_3S_4-S_3R_4)-W_3(R_2S_4-S_2R_4)+$$
$$+W_4(R_2S_3-S_2R_3)]+a_7\xi_1[U_1(R_3S_4-S_3R_4)-U_3(R_1S_4-S_1R_4)+$$
$$+U_4(R_1S_3-S_1R_3)]\,, \tag{69}$$

$$\Delta_3=(\phi(t,z)+F+\xi_5)[W_1(R_2S_4-S_2R_4)-W_2(R_1S_4-S_1R_4)+$$
$$+W_4(R_1S_2-S_1R_2)]+a_7\xi_1[U_1(R_3S_4-S_3R_4)-U_3(R_1S_4-S_1R_4)+$$
$$+U_4(R_1S_3-S_1R_3)]\,, \tag{70}$$

$$\Delta_4=(\phi(t,z)+F+\xi_5)[W_1(R_2S_3-S_2R_3)-W_2(R_1S_3-S_1R_3)-$$
$$-W_3(R_1S_2-S_1R_2)]+a_7\xi_1[U_1(R_2S_3-S_2R_3)-U_2(R_1S_3-S_1R_3)-$$
$$-U_3(R_1S_2-S_1R_2)]\,. \tag{71}$$

NUMERICAL RESULTS

The analytical results have been verified with the help of numerical example by taking wood slab as a porous material. The physical constants for the material are given by Chang and Weng /44/, and Yang et. al /45/: λ_0 = 46.92×10^9 N/m^2; μ_0 = 24.17×10^9 N/m^2; ρ_0 = 370 kg/m^3; ν_0 = 0.33; m_0 = 10 %; α^m = 2.68×10^{-3} cm/cm(% H_2O); T_0 = 283 °K; α^T = 31.3×10^{-6} cm/cm(°K); k_0 = 0.65 W/m(°K); c = 2500 J/kg(°K); k_{m0} = 2.2×10^{-8} kg/msM; D_m = 2.16×10^{-6} m^2/s; $D_T = k/\rho_0 c$; D_m^T = 0.648×10^{-6} m^2(% H_2O)/s(°K); D_T^m = 2.1×10^{-7} m^2(°K)/s(% H_2O).

The numerical results have been carried out on the surface z = 1.0; t = 0.5, and dimensionless parameter l = 1.0. The graphical results of displacement, stress, moisture concentration and temperature field are shown in Figs. 1 to 6 for two values of initial stress parameters P_0 = 1, 10, and three values of non-homogeneous parameter n = 9, 0.5, 1.0.

DISCUSSION

The variations of tangential and normal displacement for different values of non-homogeneous parameter n and initial stress parameter P_0 are quite significant in the range $0 \le x \le 2.0$. These variations are similar in nature in the range $2.0 \le x \le 20.0$. Of these two quantities, the values of normal displacement are very close to each other and follow a linear path in the range $2.0 \le x \le 20.0$. These variations of tangential and normal displacement are shown in Figs. 1 and 2. It is observed from Figs. 3 and 4 that the variations of tangential and normal force stress are opposite in nature in the range $0 \le x \le 6.0$. While the values of tangential force stress increase sharply in this range, the values of normal force stress decrease in the same range. In the remaining range, i.e., $6.0 \le x \le 20.0$, the magnitude of the values of mechanical stress is very less and hence the variation shows a linear trend. The variation of moisture concentration for a non-homogeneous hygrothermoelastic medium with hydrostatic initial stress is more oscillatory in nature. These values of moisture concentration for a homogeneous thermoelastic medium lie in a short range, as shown in Fig. 5. Figure 6 depicts that the variations of temperature field are similar variations obtained for tangential displacement, but with difference in magnitude.

CONCLUSIONS

The non-homogeneous parameter and initial stress parameter has a significant effect on all the physical variables.

The variations of tangential and normal force stress are opposite in nature in the initial range.

The values of all the variables decrease with increase in horizontal distance.

The variations of all the quantities are more oscillatory in nature for a non-homogeneous medium.

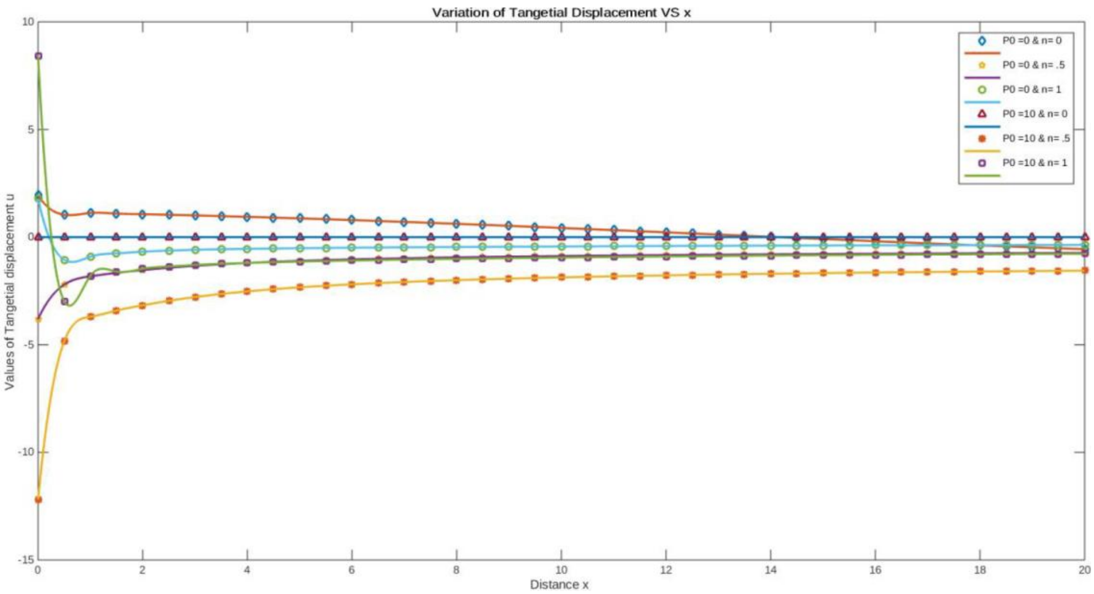

Figure 1. Variation of tangential displacement u with horizontal distance.

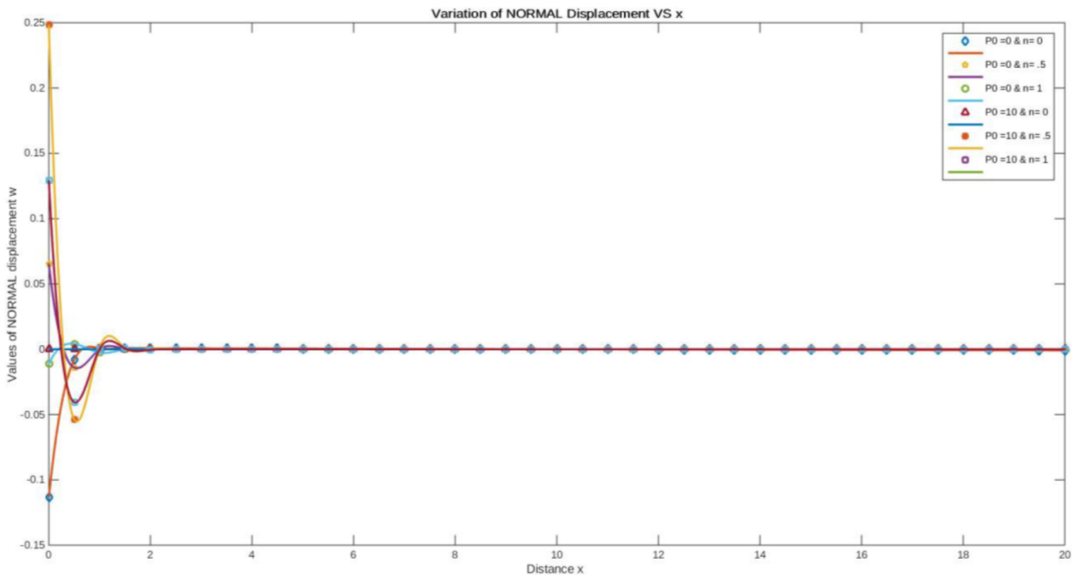

Figure 2. Variation of normal displacement w with horizontal distance.

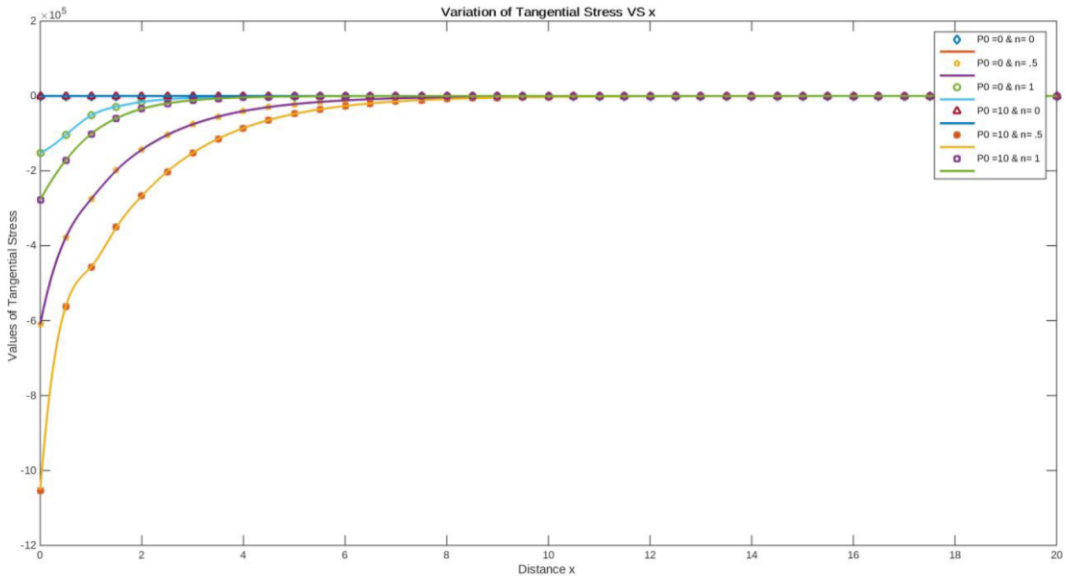

Figure 3. Variation of tangential stress σ_{zx} with horizontal distance.

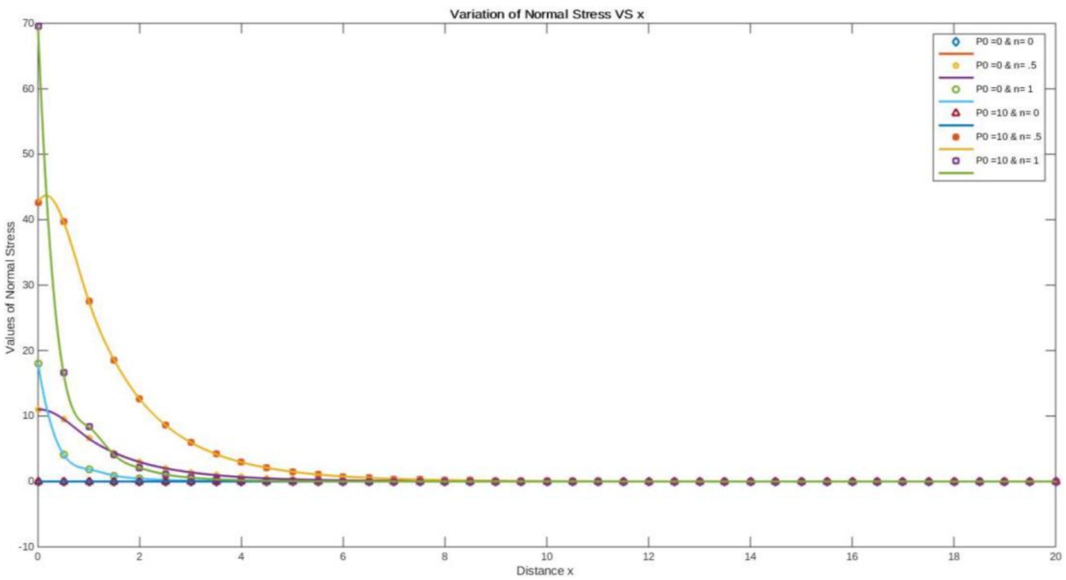

Figure 4. Variation of normal stress σ_{zz} with horizontal distance.

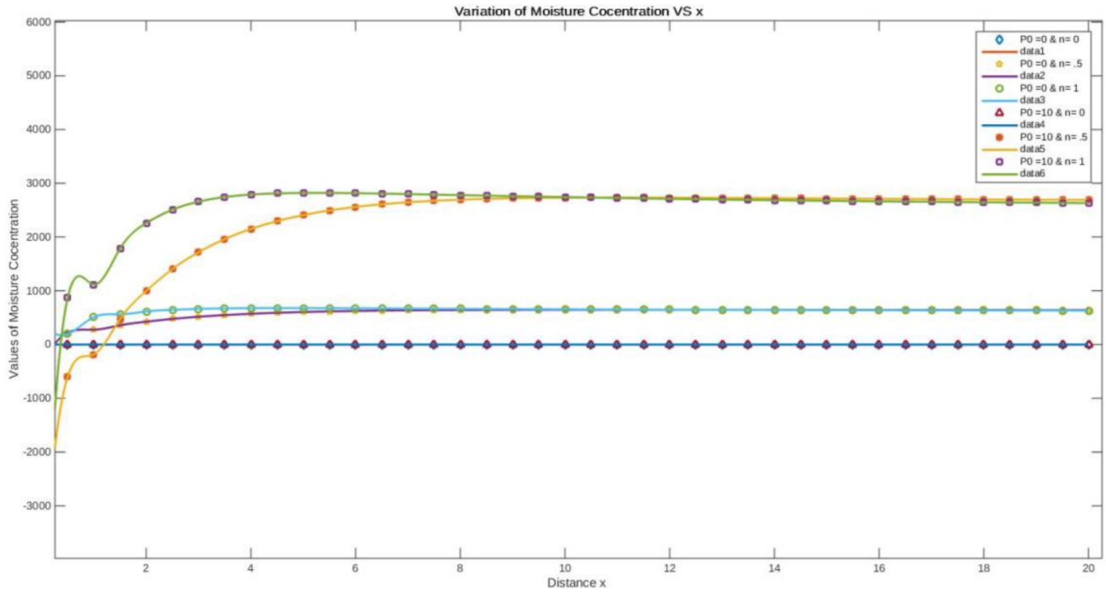

Figure 5. Variation of moisture concentration *m* with horizontal distance.

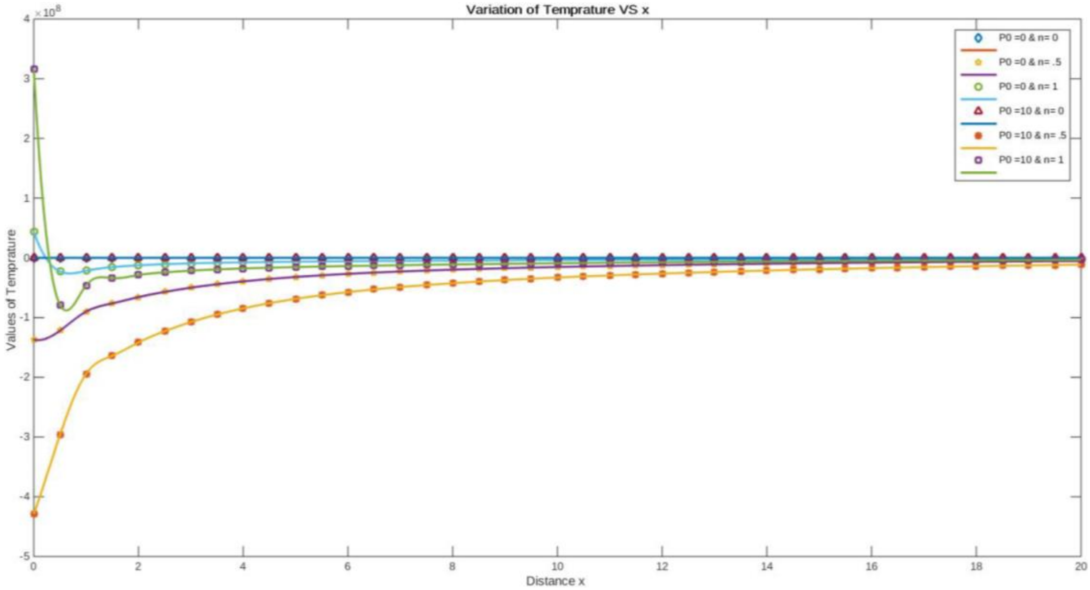

Figure 6. Variation of temperature *T* with horizontal distance.

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Appendix A

$$f_1=\frac{t_2t_{30}+t_1t_{32}+a_2t_{25}-2a_2t_{24}}{t_1t_{30}+a_2t_{24}},$$

$$f_2=\frac{n^2t_2t_{31}-t_3t_{30}+t_1t_{32}+a_2t_{26}-2n^2a_2t_{25}+t_4t_{24}}{t_1t_{30}+a_2t_{24}},$$

$$f_3=\frac{t_2t_{32}-t_3t_{31}+t_1t_{33}+a_2t_{27}-2a_2t_{26}+t_4t_{25}+t_5t_{24}}{t_1t_{30}+a_2t_{24}},$$

$$f_4=\frac{t_1t_{34}-t_3t_{32}+n^2t_2t_{33}+a_2t_{28}-2n^2a_2t_{27}+t_4t_{26}+n^2t_5t_{25}}{t_1t_{30}+a_2t_{24}},$$

$$f_5=\frac{t_2t_{34}-t_3t_{33}+t_1t_{35}+a_2t_{29}-2a_2t_{28}+t_4t_{27}+t_5t_{26}}{t_1t_{30}+a_2t_{24}},$$

$$f_6=\frac{n^2t_2t_{35}-t_3t_{34}+t_1t_{36}-2n^2a_2t_{29}+t_4t_{28}+n^2t_5t_{27}}{t_1t_{30}+a_2t_{24}},$$

$$f_7=\frac{t_2t_{36}-t_3t_{35}+t_4t_{29}+t_5t_{28}}{t_1t_{30}+a_2t_{24}},$$

$$f_8=\frac{n^2t_5t_{29}-t_3t_{36}}{t_1t_{30}+a_2t_{24}},$$

$t_1 = 1 - a_1$, $t_2 = a_7 + a_1 - 1$, $t_3 = d_1 + n^2a_7$, $t_4 = a_1b^2 - d_2 + n^2a_2$, $t_5 = d_2 - a_5b^2$, $t_6 = a_1 + a_7$, $t_7 = n^2a_7 - a_1d_4 - d_3a_4$, $t_8 = a_7d_4$, $t_9 = n^2a_2 - d_2 - a_2d_4$, $t_{10} = d_2 + a_2d_4$, $t_{11} = d_2d_4 + d_3a_4b^2$, $t_{12} = a_8a_4 - a_3$, $t_{13} = a_3d_4 - a_8a_4b^2$, $t_{14} = a_1a_{11}$, $t_{15} = (a_1 + a_7)a_{11}$, $t_{16} = n^2a_7a_{11} - a_1a_{11}b^2 - d_5a_4$, $t_{17} = a_7a_{11}b^2$, $t_{18} = a_2a_{11}$, $t_{19} = n^2a_2a_{11} - d_2a_{11} - a_2b^2a_{11}$, $t_{20} = a_{11}d_2 + a_2b^2a_{11}$, $t_{21} = d_2b^2a_{11} + d_5b^2a_4$, $t_{22} = a_4 - a_3a_{11}$, $t_{23} = a_3b^2a_{11} - a_4d_6$, $t_{24} = a_1t_{22} - t_{14}t_{12}$, $t_{25} = t_{12}(t_{15} + t_{14}) - t_{22}(t_6 + a_1)$, $t_{26} = t_{22}(t_7 + n^2t_6) + a_1t_{23} - t_{12}(t_{16} + n^2t_{15}) - t_{13}t_{14}$, $t_{27} = t_{22}(t_8 - t_7) - t_6t_{23} + t_{12}(t_{16} - t_{17}) + t_{13}t_{15}$, $t_{28} = t_7t_{23} + n^2(t_{17}t_{12} - t_8t_{22}) - t_{16}t_{13}$, $t_{29} = t_8t_{23} - t_{17}t_{13}$, $t_{30} = a_2t_{22} - t_{12}t_{18}$, $t_{31} = 3(-a_2t_{22} + t_{12}t_{18})$, $t_{32} = t_{22}(t_9 + 2n^2a_2) + a_2t_{23} - t_{12}(t_{19} + 2n^2t_{18}) - t_{18}t_{13}$, $t_{33} = t_{22}(t_{10} - t_9) - 2a_2t_{23} + t_{12}(t_{19} - t_{20}) + 2t_{18}t_{13}$, $t_{34} = t_{22}(t_{11} - n^2t_{10}) + t_9t_{23} + t_{12}(n^2t_{20} - t_{21}) - t_{19}t_{13}$, $t_{35} = t_{10}t_{23} - t_{11}t_{22} + t_{21}t_{12} - t_{20}t_{13}$, $t_{36} = t_{11}t_{23} - t_{21}t_{13}$.