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FROBENIUS SERIES SOLUTION FOR FUNCTIONALLY GRADED MATERIAL WITH EXPONENTIALLY VARIABLE THICKNESS AND MODULI

REŠENJE OBLIKA FROBENIUSOVOG REDA ZA FUNKCIONALNI MATERIJAL EKSPONENCIJALNO PROMENLJIVE DEBLJINE I MODULA ELASTIČNOSTI

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Keywords
- power series method
- elastic moduli
- thick-walled cylinder
- internal pressure

Abstract

In this paper, a Frobenius series solution is obtained for a functionally graded non-rotating cylinder following the exponential law variation in material properties across radii. The plane strain condition is considered in which the strain along the axial direction is taken as zero. The expressions are obtained for stresses - radial and circumferential. The strains are also obtained for functionally graded material considering the problem as axi-symmetric. The expressions for the homogeneous case are obtained by making the material index zero. Graphs are plotted for stresses, strains, and displacements for the homogeneous case and are numerically discussed. The results are obtained under internal pressure in which the external pressure is kept as zero. It is seen that the radial stress is compressive at the internal radii and moves towards zero at the outer radii. The circumferential stress is tensile and is maximum at the internal radii and minimum at the outer radii.

Ključne reči
- metoda stepenog reda
- moduli elastičnosti
- debelozidi cilindar
- unutrašnji pritisak

Izvod

U ovom radu, rešenje oblika Frobeniusovog reda se dobija za nerotirajući cilindar od funkcionalnog materijala, a shodno promeni osobina materijala duž radijusa prema eksponencijalnom zakonu. Razmatra se ravno stanje deformacija, gde su deformacije u aksijalnom pravcu jednake nuli. Dobijeni su izrazi za napone - radijalne i obimske. Dobijene su i deformacije za funkcionalni materijal u uslovima osnosimetričnog problema. Izrazi za homogeni slučaj su dobijeni pod uslovom da je indeks materijala jednak nuli. Iscrtani su dijagrami za napone, deformacije i pomeranja za homogeni slučaj, a data je i diskusija numeričkih rezultata. Dobijeni su rezultati za unutrašnji pritisak kada je spoljni pritisak jednak nuli. Uočava se da su radijalni naponi – pritisni na unutrašnjem radijusu, koji se približava nuli na spoljnjem radijusu. Obimski napon je zatezni i dostiže maksimum na unutrašnjem radijusu, a minimum na spoljnjem radijusu.

INTRODUCTION

In the modern era of technology, materials play a very important role. The materials are so important that in the development of civilization, we even associate ages with them such as Stone Age, Bronze Age, etc. The materials in general are classified according to their properties and their usage such as metals, semiconductor, ceramics, etc. The combination of these individual materials can be made, called as composite materials, so that the strength of the structure can be improved, and their material cost can be saved, because it is not necessary that the same material is used throughout the structure to maintain their strength. The composite materials have been used at various places such as in the aviation sector, military equipment, transportation industry, construction sector, marine industry for construction of naval structures and many more. But a drawback of using composite materials is the delamination which is a major cause of failure of the equipment. So, a new composite material is developed by Japanese scientist in the 1980's which encompasses enhanced properties of composite materials. Such materials were first used by Japanese scientists to build a thermal barrier in a space-plane project /1/, which lead to the concept of functionally graded materials (FGMs). FGM is a material in which the composition and structure continuously vary over the dimension of the material body and hence resulting in corresponding change in properties (mechanical, thermal, electrical, etc.).

A lot of books on elasticity /2-3/ and study of behaviour of composite materials /4/ are published that helps in understanding the concept and development of theory. The processing techniques for FGMs are described in detail by Kieback et al. /5/ in 2003, and Gasik /6/ in 2010. Since its initial development for thermal barrier, its uses have grown to various fields such as sensor and energy applications /7/, thick-walled circular cylinder /8/, rotating disk /9-10/, etc. In 2016, Sahni et al. /11/ have studied the creep behaviour of SiC_p exponential volume reinforcement in FGM composite rotating cylinders. In 2018 /12/ an elastic-plastic analysis with variation in Young's modulus is done. Nejad et al. /13/ solved the problem on thick cylindrical pressure vessels using both finite element method and power series solution method and have shown the solutions are in good agreement with each other. A recently published paper in 2019 by Sandeep et al. /14/ on two-dimensional mechanical stresses for a pressurized cylinder have calculated stresses for a functionally graded material. In 2019, Parth et al. /15/ analysed a thick-walled cylinder with inner layer of FGM and the outer composite layer. Recent work on functionally graded materials related to disc and cylinder is done by Sahni and Mehta in research papers, /16–17/.

In this paper, the problem of a thick-walled cylindrical pressure vessel is considered without body force, in which the Young's modulus and thickness are varying exponentially. The Poisson ratio is taken as constant and the cylinder is non-rotating. Expressions for stresses, strains and displacements are calculated. The graphs are plotted for homogeneous case by making the materials index as zero.

GOVERNING EQUATIONS

The governing equations are basic equations which need to be considered to study the behaviour of mechanical deformations under external actions. So, to study any mechanical deformation, the three basic governing equations are to be considered, i.e. firstly the equilibrium equation (i.e. the sum of body and surface forces is zero); secondly, the relation between the strain and displacement; and thirdly, the stress - strain relation (also called Hooke's law).

General form of the equilibrium equation is given as /9/,

$$\frac{d}{dr}(hr\sigma_{rr})-h\sigma_{\theta\theta}+h\rho\omega^2r^2=0\,, \tag{1}$$

where: h is thickness of the cylindrical pressure vessel; r is the radii; ω is angular speed; ρ is density; and σ_{rr}, $\sigma_{\theta\theta}$ are the stresses along radial and tangential direction, in respect.

Under absence of body forces, the Eq.(1) reduces to

$$\frac{d}{dr}(hr\sigma_{rr})-h\sigma_{\theta\theta}=0\,. \tag{2}$$

The strain displacement relations are written as, /9/,

$$\varepsilon_{rr}=\frac{du}{dr} \quad \text{and} \quad \varepsilon_{\theta\theta}=\frac{u}{r}\,, \tag{3}$$

where: u is the radial displacement; and ε_{rr}, $\varepsilon_{\theta\theta}$ are strains along radial and circumferential directions, respectively.

The strain compatibility equation using Eqs.(3) is written

$$\varepsilon_{rr}=\frac{d}{dr}(r\varepsilon_{\theta\theta})\,. \tag{4}$$

Hooke's law (i.e. stress-strain relation) is written as, /9/,

$$\sigma_{rr}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{rr}+\frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{\theta\theta}\,, \quad \text{and}$$

$$\sigma_{\theta\theta}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{\theta\theta}+\frac{E\nu}{(1+\nu)(1-2\nu)}\varepsilon_{rr}\,, \tag{5}$$

where: ν is Poisson's ratio; and E is Young's modulus.

MATHEMATICAL FORMULATIONS

As the cylindrical pressure vessel is made of functionally graded material, hence the Young's modulus and thickness along the wall vary from inside to outside as

$$E=E_0e^{-n\left(\frac{r}{b}\right)} \quad \text{and} \quad h=h_0e^{-m\left(\frac{r}{b}\right)}\,. \tag{6}$$

Here n and m are the index parameters corresponding to Young's modulus (E), and thickness (h), respectively.

Substituting Eqs.(3), (5) and (6) in Eq.(2), we get

$$r^2u''+ru'\left(1-\frac{r(n+m)}{b}\right)-\left(\frac{\nu_1(n+m)}{b}r+1\right)=0\,, \tag{7}$$

where: $\nu_1=\frac{\nu}{(1-\nu)}$.

The singular point of the above differential equation Eq.(7) is $r = 0$. The singular point is regular as

$$\lim_{r\to 0}(r-0)\left[\frac{r\left(1-\frac{r(n+m)}{b}\right)}{r^2}\right]=1\,, \quad \text{and}$$

$$\lim_{r\to 0}(r-0)\left[-\frac{\left(\frac{\nu_1(n+m)}{b}r+1\right)}{r^2}\right]=-1\,.$$

Both limits are finite. Hence the solution exists and the Frobenius method can be used to find the solution of the differential equation Eq.(7).

Now, assuming the solution in the power series form as

$$u=\sum_{i=0}^{\infty}a_ir^{i+s}\,,$$

we need to find u' and u'' from the above power series, and hence

$$u'=\sum_{i=0}^{\infty}(i+s)a_ir^{i+s-1}\,, \quad \text{and}$$

$$u''=\sum_{i=0}^{\infty}(i+s)(i+s-1)a_ir^{i+s-2}\,.$$

Substituting u, u', u'' in Eq.(7), we get

$$\sum_{i=0}^{\infty}a_ir^{i+s}\left\{(i+s)(i+s-1)+(i+s)-1\right\}-$$

$$-\sum_{i=0}^{\infty}a_ir^{i+s+1}\left\{\frac{(n+m)(i+s)}{b}+\frac{\nu_1(n+m)}{b}\right\}=0\,. \tag{8}$$

Comparing the coefficients of the lowest degree term (r^s), i.e. by putting $i = 0$ in Eq.(8), we get $s = 1, -1$. Thus we get the roots as distinct and differing by an integer, and hence, the solution is given as

$$u=C_1(u)_{s=1}+C_2\left(\frac{\partial u}{\partial s}\right)_{s=-1}\,. \tag{9}$$

The general recurrence relation is formed by comparing the next higher term, i.e. from (r^{s+1}) onwards,

$$a_{i+1}=\frac{\left\{(n+m)(i+s)+\nu_1(n+m)\right\}}{b\left\{(s+i+1)^2-1\right\}}a_i\,, \quad i\ge 0\,. \tag{10}$$

Thus, for s = 1, the above recurrence relations become

$$a_{i+1}=\frac{\{(n+m)(i+1)+\nu_1(n+m)\}}{b\{(i+2)^2-1\}}a_i,\quad i\geq 0. \qquad (11)$$

Hence, one of the solutions is given as

$$u(s=1)=u_1=s\left(\sum_{i=0}^{\infty}a_i r^i\right),\quad a_0\neq 0, \qquad (12)$$

and the second solution is defined as

$$u(s=-1)=u_2=\left(\frac{\partial u}{\partial s}\right)_{s=-1}.$$

In general, the constants a_{j+1} for (s = 1) are calculated using the recurrence relation as

$$a_{j+1}=\frac{(n+m)^{j+1}(\nu_1+1)(\nu_1+2)(\nu_1+3)-(\nu_1+j+1)}{b^{j+1}\prod_{i=0}^{j}\{(i+2)^2-1\}}a_0,\quad j\geq 0 \qquad (13)$$

Thus,

$$u_1=r(a_0+a_1r+a_2r^2+\ldots)=a_0r\left(1+\frac{(n+m)(\nu_1+1)}{3b}r+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{8\cdot 3b^2}r^2+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{15\cdot 8\cdot 3b^3}r^3+\ldots\right), \qquad (14)$$

and

$$u_2=\left(\frac{\partial u}{\partial s}\right)_{s=-1}=a_0r^{-1}\left[\left\{-\frac{(n+m)(\nu_1-1)\nu_1}{2b^2}r^2\log(r)+\ldots\right\}+\left\{1-\frac{(n+m)}{b}r+\frac{(n+m)^2}{2b^2}\left(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_2-3)(\nu_1-2)}{12}\right)r^2+\ldots\right\}\right] \qquad (15)$$

Hence, the complete solution is calculated from Eq.(9), given as

$$u=C_1u_1+C_2u_2. \qquad (16)$$

In an expanded form it is written as

$$u=A_1r\left(1+\frac{(n+m)(\nu_1+1)}{3b}r+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{8\cdot 3b^2}r^2+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{15\cdot 8\cdot 3b^2}r^3+\ldots\right)+A_2r^{-1}\left[\left\{-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}\times\right.\right.$$
$$\left.\left.\times r^2\log(r)+\ldots\right\}+\left\{1-\frac{(n+m)}{b}r+\frac{(n+m)^2}{2b^2}\left(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\right)r^2+\ldots\right\}\right], \qquad (17)$$

where: $A_1 = C_1a_0$; and $A_2 = C_2a_0$.

The strains-radial and circumferential are calculated from Eqs.(3).

Radial strain is calculated as

$$\varepsilon_{rr}=\frac{du}{dr}=A_1\left(1+\frac{(n+m)(\nu_1+1)}{3b}r+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{8\cdot 3b^2}r^2+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{15\cdot 8\cdot 3b^3}r^3+\ldots\right)+$$
$$+A_1r\left(\frac{(n+m)(\nu_1+1)}{3b}+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{12b^2}r+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{120b^3}r^2+\ldots\right)-A_2r^{-2}\times$$
$$\times\left[\left\{-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r^2\log(r)+\ldots\right\}+\left\{1-\frac{(n+m)}{b}r+\frac{(n+m)^2}{2b^2}\left(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\right)r^2+\ldots\right\}\right]+$$
$$+A_2r^{-1}\left[\left\{-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r-\frac{(n+m)^2(\nu_1-1)\nu_1}{b^2}r\log(r)+\ldots\right\}+\left\{-\frac{(n+m)}{b}+\frac{(n+m)^2}{2b^2}\times\right.\right.$$
$$\left.\left.\times\left(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\right)r+\ldots\right\}\right], \qquad (18)$$

and circumferential strain is calculated as

$$\varepsilon_{\theta\theta}=\frac{u}{r}=A_1\left(1+\frac{(n+m)(\nu_1+1)}{3b}r+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{8\cdot 3b^2}r^2+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{15\cdot 8\cdot 3b^3}r^3+\ldots\right)+A_2r^{-2}\times$$
$$\times\left[\left\{-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r^2\log(r)+\ldots\right\}+\left\{1-\frac{(n+m)}{b}r+\frac{(n+m)^2}{2b^2}\left(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\right)r^2+\ldots\right\}\right]. \qquad (19)$$

Now, substituting the radial and circumferential strain obtained above in Eq.(5), we get the radial and circumferential stresses as

$$\sigma_{rr}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[A_1\left(1+\frac{(n+m)(\nu_1+1)}{3b}r+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{8\cdot 3b^2}r^2+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{15\cdot 8\cdot 3b^3}r^3+\ldots\right)+A_1r\times\right.$$

$$\times\Bigg(\frac{(n+m)(\nu_1+1)}{3b}+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{12b^2}r+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{120b^3}r^2+\ldots\Bigg)-A_2r^{-2}\Bigg[\Bigg[\Bigg\{-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r^2\log(r)+\ldots\Bigg\}+$$

$$+\Bigg\{1-\frac{(n+m)}{b}r+\frac{(n+m)^2}{2b^2}\Bigg(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\Bigg)r^2+\ldots\Bigg\}\Bigg]\Bigg]+A_2r^{-1}\Bigg[\Bigg[\Bigg\{-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r-$$

$$-\frac{(n+m)^2(\nu_1-1)\nu_1}{b^2}r\log(r)+\ldots\Bigg\}+\Bigg\{-\frac{(n+m)}{b}+\frac{(n+m)^2}{2b^2}\Bigg(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\Bigg)r+\ldots\Bigg\}\Bigg]\Bigg]+$$

$$+\frac{E\nu}{(1+\nu)(1-2\nu)}\Bigg[A_1\Bigg(1+\frac{(n+m)(\nu_1+1)}{3b}r+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{8\cdot 3b^2}r^2+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{15\cdot 8\cdot 3b^3}r^3+\ldots\Bigg)+A_2r^{-2}\times$$

$$\Bigg[\Bigg[\Bigg(-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r^2\log(r)+\ldots\Bigg)+\Bigg\{1-\frac{(n+m)}{b}r+\frac{(n+m)^2}{2b^2}\Bigg(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\Bigg)r^2+\ldots\Bigg\}\Bigg]\Bigg]\Bigg], \quad (20)$$

and

$$\sigma_{\theta\theta}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\Bigg[A_1\Bigg(1+\frac{(n+m)(\nu_1+1)}{3b}r+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{8\cdot 3b^2}r^2+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{15\cdot 8\cdot 3b^3}r^3+\ldots\Bigg)+A_2r^{-2}\times$$

$$\times\Bigg[\Bigg\{-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r^2\log(r)+\ldots\Bigg\}++\Bigg\{1-\frac{(n+m)}{b}r+\frac{(n+m)^2}{2b^2}\Bigg(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\Bigg)r^2+\ldots\Bigg\}\Bigg]\Bigg]+$$

$$+\frac{E\nu}{(1+\nu)(1-2\nu)}\Bigg[A_1\Bigg(1+\frac{(n+m)(\nu_1+1)}{3b}r+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{8\cdot 3b^2}r^2+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{15\cdot 8\cdot 3b^3}r^3+\ldots\Bigg)+A_1r\times$$

$$\times\Bigg(\frac{(n+m)(\nu_1+1)}{3b}+\frac{(n+m)^2(\nu_1+1)(\nu_1+2)}{12b^2}r+\frac{(n+m)^3(\nu_1+1)(\nu_1+2)(\nu_1+3)}{120b^3}r^2+\ldots\Bigg)-A_2r^{-2}\Bigg[\Bigg[\Bigg(-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r^2\log(r)+\ldots\Bigg)+$$

$$+\Bigg\{1-\frac{(n+m)}{b}r+\frac{(n+m)^2}{2b^2}\Bigg(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\Bigg)r^2+\ldots\Bigg\}\Bigg]\Bigg]+A_2r^{-1}\Bigg[\Bigg[\Bigg\{-\frac{(n+m)^2(\nu_1-1)\nu_1}{2b^2}r-$$

$$-\frac{(n+m)^2(\nu_1-1)\nu_1}{b^2}r\log(r)+\ldots\Bigg\}+\Bigg\{-\frac{(n+m)}{b}+\frac{(n+m)^2}{b^2}\Bigg(-\frac{\nu_1(\nu_1+1)}{6}-\frac{(2-\nu_1)(\nu_1-1)}{2}-\frac{(\nu_1-3)(\nu_1-2)}{12}\Bigg)r+\ldots\Bigg\}\Bigg]\Bigg]\Bigg]. \quad (21)$$

For homogeneous case ($m = n = 0$), differential equations Eq.(7) reduce to

$$r^2u''+ru'-u=0\,. \quad (22)$$

The above is a Cauchy-Euler second order ordinary differential equation and its analytical solution is given as

$$u=C_3r+\frac{C_4}{r}\,. \quad (23)$$

The boundary conditions are defined as

$$\sigma_{rr}=-p_1 \quad \text{at} \quad r=a, \quad \text{and} \quad \sigma_{rr}=-p_2 \quad \text{at} \quad r=b\,, \quad (24)$$

where: p_1 and p_2 are the pressure at internal and external radii of the cylinder, respectively.

The strains are now expressed as

$$\varepsilon_{rr}=C_3-\frac{C_4}{r^2} \quad \text{and} \quad \varepsilon_{\theta\theta}=C_3+\frac{C_4}{r^2}\,. \quad (25)$$

Thus, the stresses are now expressed as

$$\sigma_{rr}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(C_3-\frac{C_4}{r^2}\right)+\frac{E\nu}{(1+\nu)(1-2\nu)}\left(C_3+\frac{C_4}{r^2}\right)$$

and

$$\sigma_{\theta\theta}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(C_3+\frac{C_4}{r^2}\right)+\frac{E\nu}{(1+\nu)(1-2\nu)}\left(C_3-\frac{C_4}{r^2}\right). \quad (26)$$

Now using the boundary conditions Eq.(24) on the above Eqs.(26), we get the constants as

$$-p_1=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(C_3-\frac{C_4}{a^2}\right)+\frac{E\nu}{(1+\nu)(1-2\nu)}\left(C_3+\frac{C_4}{a^2}\right),$$

$$-p_2=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left(C_3-\frac{C_4}{b^2}\right)+\frac{E\nu}{(1+\nu)(1-2\nu)}\left(C_3+\frac{C_4}{b^2}\right), \quad (27)$$

where: C_3 and C_4 are constants of integration which can be found from Eqs.(27):

$$C_3=\frac{(1+\nu)(-1+2\nu)(a^2p_1-b^2p_2)}{(a^2-b^2)E}\,, \quad \text{and}$$

$$C_4=-\frac{a^2b^2(1+\nu)(p_1-p_2)}{(a^2-b^2)E}\,. \quad (28)$$

Thus putting the values of C_3 and C_4, we are able to find the stresses, strains, and displacement for the homogeneous case by using Eqs.(26), (25) and (23), respectively.

Using the Frobenius method, we can get the solutions of stresses and strains from equations (18), (19), (20) and (21), respectively.

NUMERICAL DISCUSSION

To analyse the effect of gradient parameter, the arbitrary values for the analysis are taken as m, n = 0; Poisson ratio as ν = 0.3; Young's modulus E_0 = 200 MPa; internal and external radii of the cylinder a = 0.2 m, b = 0.5 m; h_0 = 10 cm; and pressures p_1 = 100 MPa, and p_2 = 0 MPa. In Figs. 1 and 2, the Young's modulus and thickness are plotted against radii.

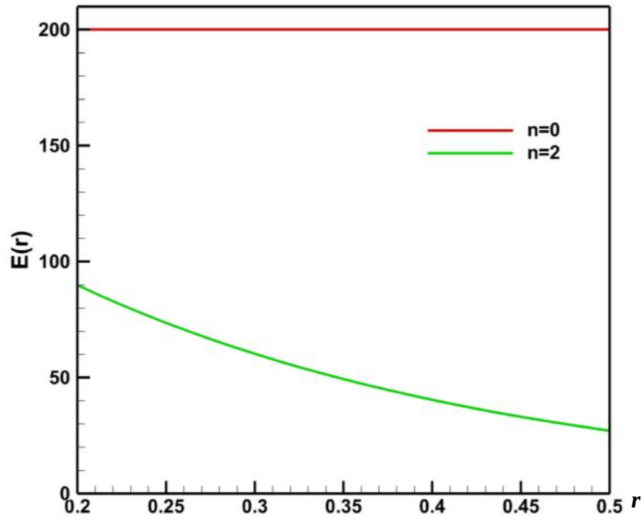

Figure 1. Young's modulus variation against radii for homogeneous and non-homogeneous case.

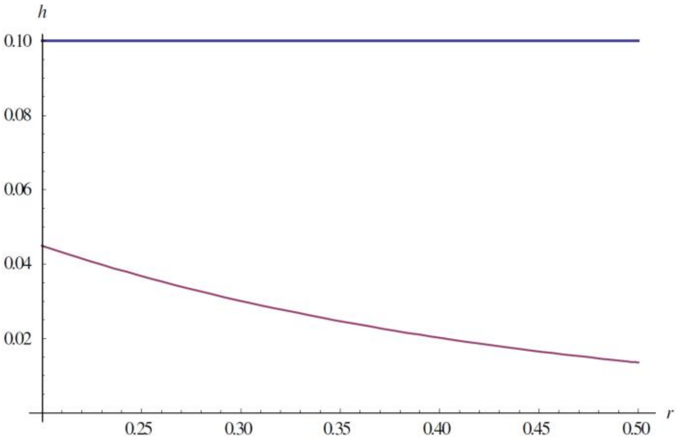

Figure 2. Thickness variation against radii for homogeneous and non-homogeneous case.

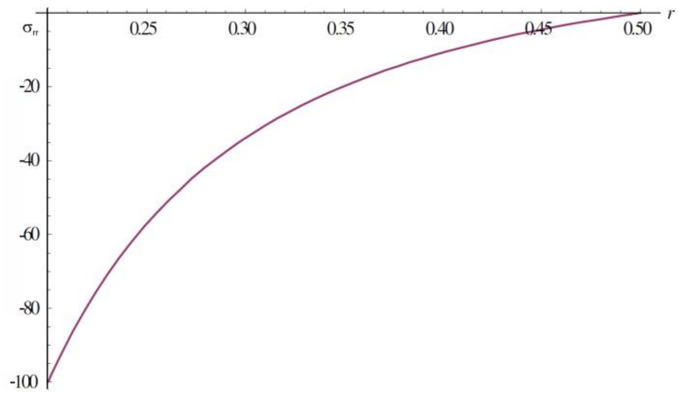

Figure 3. Radial stress vs. radii for homogeneous case under internal pressure p_1 = 10 MPa.

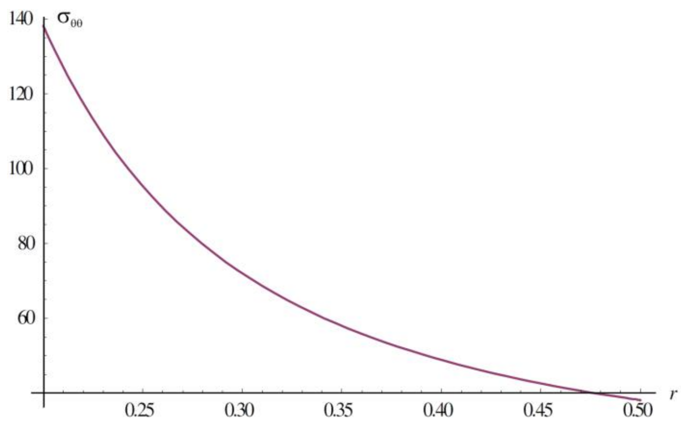

Figure 4. Circumferential stress for homogeneous case under internal pressure p_1 = 10 MPa.

The radial and circumferential stresses are plotted against radii for homogeneous case, i.e. when m, n = 0, as shown in Figs. 3 and 4, respectively.

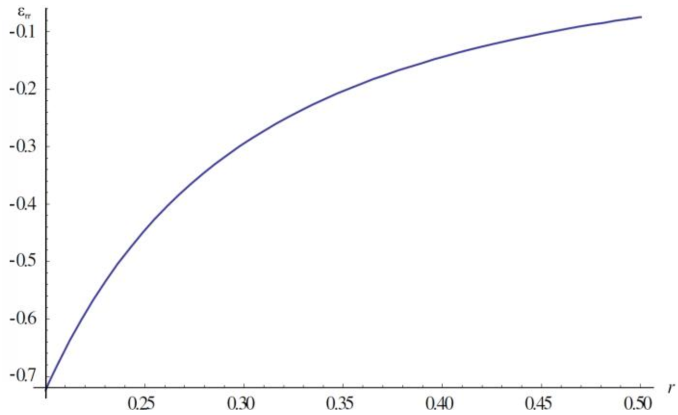

Figure 5. Radial strain vs. radii for homogeneous case under internal pressure p_1 = 10 MPa.

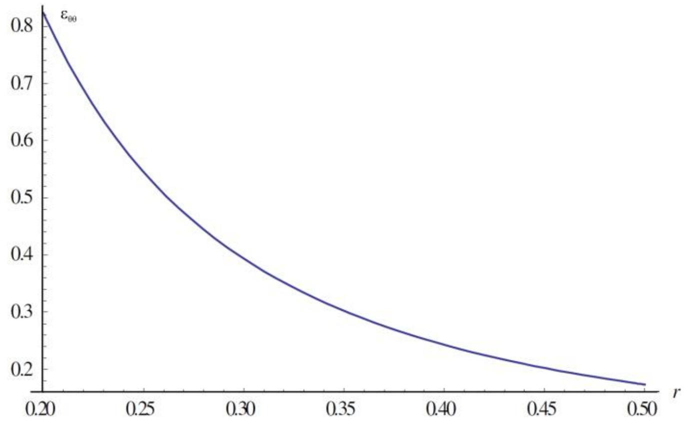

Figure 6. Circumferential strain vs. radii for homogeneous case under internal pressure p_1 = 10 MPa.

It is seen that the radial stress is less as compared to circumferential stress. The circumferential stress acts as a resisting stress and the radial stress tries to push the radii points outside, which the circumferential stress resists. It is seen that the radial stress is compressive at internal radii and moves towards zero at the outer radii. The circumferential stress is tensile and is maximum at the internal radii and minimum at the outer radii. The radial and circumferential strain are decreasing when we move from internal to external radii. It is seen that radial strain is decreasing from internal to external radii, which is compressive, whereas circumferential strain is maximum at internal radii and minimum at external surface. The radial and circumferential strain are depicted graphically in Figs. 5 and 6, respectively.

The radial displacement is also calculated for homogeneous case and is shown in Fig. 7, which also decreases from internal to external surface.

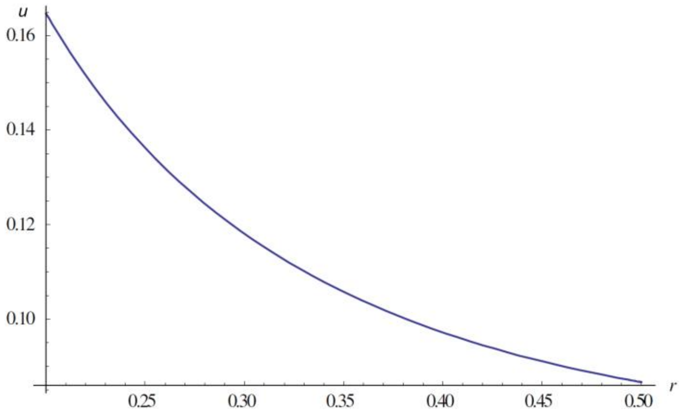

Figure 7. Radial displacement vs. radii for homogeneous case under internal pressure p_1 = 10 MPa.

CONCLUSIONS

In this paper, the stresses, strains, and displacement are calculated for both homogeneous and non-homogeneous case using power series method. The expressions are calculated for homogeneous case and graphs are plotted. It is seen that the radial stress is high as compared to the circumferential stresses at the inner surface, and it decreases at the outer surface and vanishes thus preventing the cylinder to move out and cause fracture. With the vanishing of stresses at the outer radii, the radial displacement is decreasing from internal to external radii. The strains also show a decrease from internal to external radius which is also true when physically verified.

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

Nomenclature

h	thickness (cm)
r	radii (m)
σ_{rr}	non-dimensional radial stress
$\sigma_{\theta\theta}$	non-dimensional circumferential stress
ε_{rr}	non-dimensional radial strain
$\varepsilon_{\theta\theta}$	non-dimensional circumferantial strain
u	non-dimensional radial displacement
E	Young's modulus (MPa)
ν	Poisson's ratio
n, m	gradation parameter
C_1, C_2	arbitrary constants
p_1, p_2	internal and external pressure

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