Richa Sharma[*], *Varsha Singhal, Prani Sachan*

ELASTIC-PLASTIC AXI-SYMMETRICAL BENDING OF FUNCTIONALLY GRADED RECTANGULAR WIDE PLATES

ELASTOPLASTIČNO OSNOSIMETRIČNO SAVIJANJE PRAVOUGAONIH ŠIROKIH PLOČA OD FUNKCIONALNIH MATERIJALA

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Adresa autora / Author's address:
Department of Mathematics, Jaypee Institute of Information Technology, Noida, India
email: richa.ggit@gmail.com

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Abstract

The presented problem is based on the study of elastic plastic axi-symmetrical bending of functionally graded rectangular wide plates. The concept of classical theory is applied in the problem and by using the Tresca's yield criterion, a second order differential equation is obtained as a governing equation of the problem. The method of infinite series solution is applied to solve the governing differential equation. On the basis of the numerical values and graphs it is concluded that the bending of axi-symmetrical wide plates with non-homogeneity parameter is safe for designing as compared to wide plates, because circumferential stresses are lesser for bending plate.

Izvod

Problem koji je predstavljen je zasnovan na izučavanju elastoplastičnog osnosimetričnog savijanja širokih ploča od funkcionalnog materijala. Primenjen je koncept klasične teorije u problemu, korišćenjem kriterijuma tečenja Treska, i stoga se dobija diferencijalna jednačina drugog reda kao osnovna jednačina za rešavanje problema. Primenjuje se metoda rešenja oblika beskonačnog reda za rešavanje polazne diferencijalne jednačine. Na osnovu numeričkih rezultata i dijagrama, zaključuje se da je savijanje osnosimetričnih širokih ploča sa parametrom nehomogenosti u domenu bezbednog projektovanja, u poređenju sa savijanjem širokih ploča, zbog manjih obimskih napona pri savijanju.

INTRODUCTION

The concept of functionally graded materials was first discovered in Japan in 1984 during the project of a space plane. Functionally Graded materials (FGM's) are the materials whose properties are changed by varying the composition of the functions. The demand of the FGM is increasing due to their ability of working under high pressure and temperature. The FGM's has applications in so many areas like aerospace, nuclear reactors, medicines and so forth. In present days, rectangular plates under pressure have attracted the interest of researchers due to their wide range of industrial applications. In general, the rectangular plates which works under high temperature and pressure conditions require a strict analysis of stresses for an optimum design thus efforts are continuously made to increase the reliability of such structures. The problems on deformations of plates and shells can be found in textbooks /1-5/. Sharma /6/ analysed the creep stresses in bending of transversely isotropic rectangular plates made of functionally graded materials and observed that transversely isotropic cylinder of high functionally graded material with internal pressure and nonlinear strain measure is a better alternate for designing purpose. Sharma and Yadav /7/ investigated the stresses in a rotating cylinder made of FGM material subjected to temperature and pressure by applying finite difference method and observed that composite cylinder of FGM is better as compared to homogeneous cylinders. Sharma and Yadav /8/ analysed thermal stresses and strains in rotating annular disk by applying finite difference method with von-Mises' yield criterion and non-linear strain hardening measure and found that disk made of functionally graded material reduces the risk of fractures in the structures. Creep stresses were analysed by Sharma and Panchal /9/ in pressurized thick-walled rotating spherical shell of FGM with temperature. Shahriari et al. /10/ studied vibrations in rotating disk by using generalized differential quadrature method and analysed that use of functionally graded material could decrease the value of radial stresses and radial displacement. A computational model for the analysis of elastic-plastic and residual stresses in functionally graded rotating solid shafts has been discussed by Argeso and Eraslan /11/ using von-Mises' yield criterion, total deformation theory and Swift's nonlinear hardening law. Sharma and Radaković /12/ gave an analytic solution of plastic and transitional stresses in a thin rotating disc of piezo-electric material subjected to internal pressure and concluded that isotropic material is better than piezoelectric material.

The objective of this study is to evaluate, stresses in Elastic and plastic state are calculated for thin rotating disk of FGM by using infinite series solution method with Tresca's yield criteria.

MATHEMATICAL FORMULATION

The plate is made of isotropic material with inner radius a and outer radius b. The plane stress condition is considered in the present problem.

Equation of equilibrium is given by

$$\frac{d(T_{rr})}{dr} - T_{\theta\theta} = 0 \,. \tag{1}$$

Strain displacement relation is considered to be

$$e_r = \frac{du}{dr}\,, \quad e_\theta = \frac{u}{r}\,, \tag{2}$$

where: u is the radial displacement.

The compatibility equation is given by

$$\frac{de_\theta}{dr} + \frac{e_\theta - e_r}{r} = 0 \,. \tag{3}$$

The Tresca's yield condition is given as

$$T_{ee} = 0.5(T_{rr} + T_{\theta\theta})\,, \tag{4}$$

where: T_{rr} is radial stress; and $T_{\theta\theta}$ is circumferential stress.

The relationship between stresses and strains for isotropic materials in the elastic state are given by

$$e_r^e = \frac{[T_{rr} - \nu(T_{\theta\theta} + T_{zz})]}{E}\,,$$

$$e_\theta^e = \frac{[T_{\theta\theta} - \nu(T_{rr} + T_{zz})]}{E}\,,$$

$$e_z^e = \frac{[T_{zz} - \nu(T_{rr} + T_{\theta\theta})]}{E}\,, \tag{5}$$

where: e_r^e are elastic radial strains; e_θ^e are elastic circumferential strains; and e_z^e are elastic axial strains.

The relationships between stresses and strains for isotropic materials in the plastic state are given by

$$e_r^p = \frac{e_e^p[T_{rr} - 0.5(T_{\theta\theta} + T_{zz})]}{T_{ee}}\,,$$

$$e_\theta^p = \frac{e_e^p[T_{\theta\theta} - 0.5(T_{rr} + T_{zz})]}{T_{ee}}\,,$$

$$e_z^p = \frac{e_e^p[T_{zz} - 0.5(T_{rr} + T_{\theta\theta})]}{T_{ee}}\,, \tag{6}$$

where: T_{ee} is the equivalent stress; $e_e{}^p$, $e_\theta{}^p$, $e_r{}^p$, $e_z{}^p$ are the plastic strains.

The total radial, circumferential and axial strains are given by

$$e_r = e_r^e + e_r^p\,, \quad e_\theta = e_\theta^e + e_\theta^p\,, \quad e_z = e_z^e + e_z^p\,. \tag{7}$$

The radial, circumferential and axial stresses in terms of Airy's function are defined as

$$T_{rr} = \frac{\phi}{r}\,, \quad T_{\theta\theta} = \frac{d\phi}{dr}\,, \quad T_{zz} = 0\,. \tag{8}$$

Substituting the values of $e_r{}^e$, $e_\theta{}^e$, $e_z{}^e$ from Eq.(5) in Eq.(7) we have

$$e_r = \frac{\frac{\phi}{r} - \frac{d\phi}{dr}}{E} + e_r^p\,, \quad e_\theta = \frac{\frac{d\phi}{dr} - \nu\frac{\phi}{r}}{E} + e_\theta^p\,,$$

$$e_z = \frac{\left(\frac{d\phi}{dr} + \frac{\phi}{r}\right)}{E} + e_z^p\,. \tag{9}$$

Substituting these values of e_r and e_θ from Eq.(9) in compatibility Eq.(3), we get

$$\frac{1}{E}\left[r^2\phi'' + r\phi'\left(1 - r\frac{E'}{E}\right) + \phi\left(-r\nu' - 1 + \frac{\nu r E'}{E}\right)\right] +$$

$$+\left(\frac{de_\theta^p}{dr} + e_\theta^p - e_r^p\right) = 0\,, \tag{10}$$

where: $E = \dfrac{3}{2-C} = \dfrac{3}{2 - C_0\left(\frac{r}{b}\right)^k}$; and $\nu = \dfrac{1-C}{2-C} = \dfrac{b^k - C_0 r^k}{2b^k - C_0 r^k}$;

$E' = \dfrac{3b^k k C_0 r^{k-1}}{\left(2b^k - C_0 r^k\right)^2}$; $\nu' = \dfrac{-C_0 k b^k r^{k-1}}{\left(2b^k - C_0 r^k\right)^2}$; and variable compressibility $C = C_0\left(\dfrac{r}{b}\right)^k$.

By putting the values of E' and ν' in the Eq.(10), we get

$$r^2\phi'' + r\phi'\left(1 - \frac{kC_0\left(\frac{r}{b}\right)^k}{2 - C_0\left(\frac{r}{b}\right)^k}\right) + \phi\left(\frac{kC_0\left(\frac{r}{b}\right)^k - \frac{k}{2}\left[C_0\left(\frac{r}{b}\right)^k\right]^2}{\left[2 - C_0\left(\frac{r}{b}\right)^k\right]^2} - \right.$$

$$\left. -1 + \frac{1}{4}\frac{kC_0\left(\frac{r}{b}\right)^k}{2 - C_0\left(\frac{r}{b}\right)^k}\right) = 0\,. \tag{11}$$

Equation (11) is a differential equation of the second order that is solved by using infinite series solution method. Let the solution to Eq.(11) in terms of power series be

$$\phi = \sum_{n=0}^{\infty} a_n r^{m+n}\,. \tag{12}$$

Differentiating Eq.(12) we have

$$\phi' = \sum_{n=0}^{\infty} (m+n) a_n r^{m+n-1}$$

and

$$\phi'' = \sum_{n=0}^{\infty} (m+n)(m+n-1) a_n r^{m+n-2}\,. \tag{13}$$

Substituting the values of ϕ, ϕ', ϕ'' from Eqs.(12) and (13) into Eq.(11), we have

$$\sum_{n=0}^{\infty}\left\{(m+n)(m+n-1)\left(\frac{-4C_0}{b^k}\right) - \frac{4(1+k)C_0}{b^k}(m+n) + \frac{3C_0 k - 2C_0}{b^k}\right\}\times$$

$$\times a_n r^{m+n+k} + 8\sum_{n=0}^{\infty}\left\{(m+n)^2 - 1\right\} a_n r^{m+n} = 0\,. \tag{14}$$

Equating to zero the coefficient of lowest degree term of r to zero, by putting $n = 0$, in Eq.(14) we have

$$8(m^2 - 1)a_0 = 0\,,$$

that gives the roots of the equation as $m = \pm 1$.

Roots of the equation are real and distinct, and differ by an integer, therefore by Frobenius method of series solution the complete solution of the Eq.(11) is given by

$$\phi = (\phi)_{m=1} + \left(\frac{d\phi}{dr}\right)_{m=-1}. \tag{15}$$

The complete solution ϕ for compressibility parameter $k = 1$ is given by

$$\phi = A\left\{r + \frac{C_0}{4b}r^2 + \frac{C_0^2}{128b^2}r^3 + \frac{253C_0^3}{23040b^3}r^4 + \frac{3289C_0^4}{221184b^4}r^5 + \ldots\right\} + B\log r\left\{-\frac{C_0^2}{16b^2}r - \frac{C_0^3}{64b^3}r^2 + \frac{11C_0^4}{2048b^4}r^3 + \ldots\right\} + B\left\{\frac{2}{r} + \frac{C_0}{b} - \frac{C_0^2}{256b^2}r - \frac{C_0^3}{96b^3}r^2 + \frac{109C_0^4}{24576b^4}r^3 + \ldots\right\}. \tag{16}$$

The complete solution for $k = 2$ is given by

$$\phi = A\left\{r + \frac{C_0}{4b}r^3 + \frac{7C_0^2}{192b^4}r^4 + \ldots\right\} + B\log r\left\{-\frac{C_0}{b^2}r - \frac{C_0^2}{8b^4}r^3 + \ldots\right\} + B\left\{\frac{2}{r} - \frac{C_0^2}{32b^4}r^3 + \ldots\right\}. \tag{17}$$

Using Eqs.(8) and (16) the radial and circumferential stresses for non-homogeneity parameter $k = 1$ are given by

$$T_{rr} = \frac{1}{r}A\left\{r + \frac{C_0}{4b}r^2 + \frac{C_0^2}{128b^2}r^3 + \frac{253C_0^3}{23040b^3}r^4 + \frac{3289C_0^4}{221184b^4}r^5 + \ldots\right\} + B\log r\left\{-\frac{C_0^2}{16b^2}r - \frac{C_0^3}{64b^3}r^2 + \frac{11C_0^4}{2048b^4}r^3 + \ldots\right\} + B\left\{\frac{2}{r} + \frac{C_0}{b} - \frac{C_0^2}{256b^2}r - \frac{C_0^3}{96b^3}r^2 + \frac{109C_0^4}{24576b^4}r^3 + \ldots\right\},$$

$$T_{\theta\theta} = A\left\{1 + \frac{C_0}{2b}r + \frac{3C_0^2}{128b^2}r^2 + \frac{4*253C_0^3}{23040b^3}r^3 + \frac{5*3289C_0^4}{221184b^4}r^4 + \ldots\right\} + B\log r\left\{-\frac{C_0^2}{16b^2} - \frac{C_0^3}{32b^3}r + \frac{33C_0^4}{2048b^4}r^2 + \ldots\right\} + \frac{B}{r}\left\{-\frac{C_0^2}{16b^2}r - \frac{C_0^3}{64b^3}r^2 + \frac{11C_0^4}{2048b^4}r^3 + \ldots\right\} + B\left\{-\frac{2}{r^2} - \frac{C_0^2}{256b^2} - \frac{2C_0^3}{96b^3}r + \frac{327C_0^4}{24576b^4}r^2 + \ldots\right\} \tag{18}$$

Using Eqs.(8) and (17), the radial and circumferential stresses for non-homogeneity parameter $k = 2$ are given by

$$T_{rr} = \frac{1}{r}A\left\{r + \frac{C_0}{4b^2}r^3 + \frac{7C_0^2}{192b^4}r^4 + \ldots\right\} + B\log r\left\{-\frac{C_0}{8b^2}r - \frac{C_0^2}{8b^4}r^3 + \ldots\right\} + B\left\{\frac{2}{r} - \frac{C_0^2}{32b^4}r^3 + \ldots\right\},$$

$$T_{\theta\theta} = A\left\{1 + \frac{3C_0}{4b^2}r^2 + \frac{7C_0^2}{48b^4}r^3 + \ldots\right\} + \frac{B}{r}\left\{-\frac{C_0}{b^2} - \frac{3C_0^2}{8b^4}r^2 + \ldots\right\} + B\log r\left\{-\frac{C_0}{b^2} - \frac{3C_0^2}{8b^4}r^2 + \ldots\right\} + B\left\{-\frac{2}{r^2} - \frac{3C_0^2}{32b^4}r^2 + \ldots\right\}. \tag{19}$$

The boundary conditions are taken as

$$T_{rr} = -p_1 \quad \text{at} \quad r = a, \quad \text{and} \quad T_{rr} = -p_2 \quad \text{at} \quad r = b. \tag{20}$$

Bending moment M per unit length is given by formula

$$M = \int_a^b rT_{\theta\theta}dr. \tag{21}$$

For non-homogeneity parameter $k = 1$ with radii $a = 1$ and $b = 2$, the bending moment $M = 24.34$.
For non-homogeneity parameter $k = 2$ with radii $a = 1$ and $b = 2$, the bending moment $M = 27.38$.

NUMERICAL DISCUSSION

Figures 1 to 3 are drawn for radial and circumferential stresses with (k =1, 2), and internal and external pressures are taken as $p_1 = 5$ and $p_2 = 10$ for different radii ranging from $r = 1$ to 2. It has been noticed from Fig. 1 that for $C_0 = 0.3$, radial and circumferential stresses are compressible in nature and radial stresses attain their maximum value at external surface, but circumferential stresses attain their maximum value at internal surface for non-homogeneity parameters k =1 and $k = 2$. It is also observed from Fig. 2 that radial and circumferential stresses are compressible but maximum at internal surface for $k = 1$. As non-homogeneity parameter increases ($k = 2$) circumferential stresses are on the higher side at internal surface, also these stresses increase for $C_0 = 0.4$. Figure 3 shows that the value of radial and circumferential stresses increases for $C_0 = 0.5$.

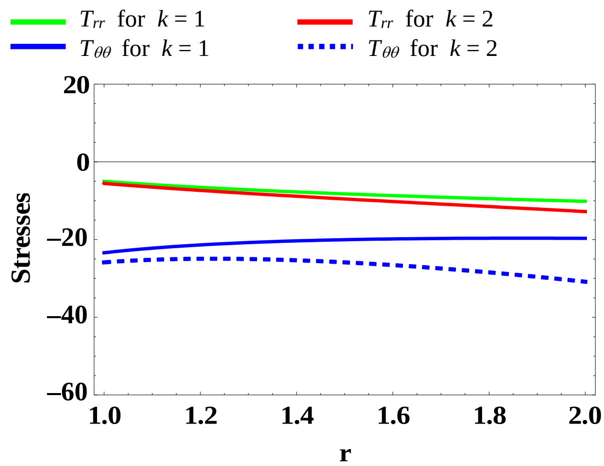

Figure 1. Radial and circumferential stresses for various radii with internal and external pressures as $p_1 = 5$ and $p_2 = 10$, and $k = 1$ and 2, and $C_0 = 0.3$.

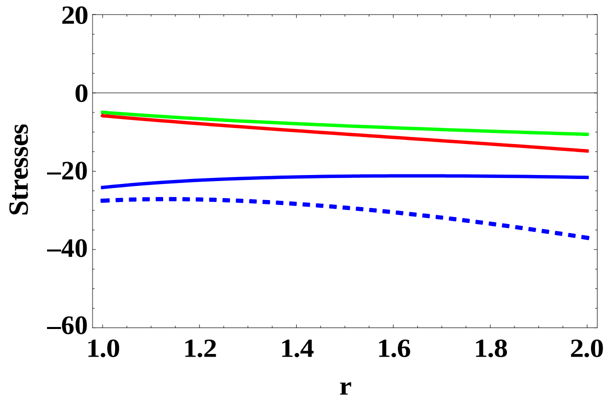

Figure 2. Radial and circumferential stresses for various radii with internal and external pressures as $p_1 = 5$ and $p_2 = 10$, and $k = 1$ and 2, and $C_0 = 0.4$.

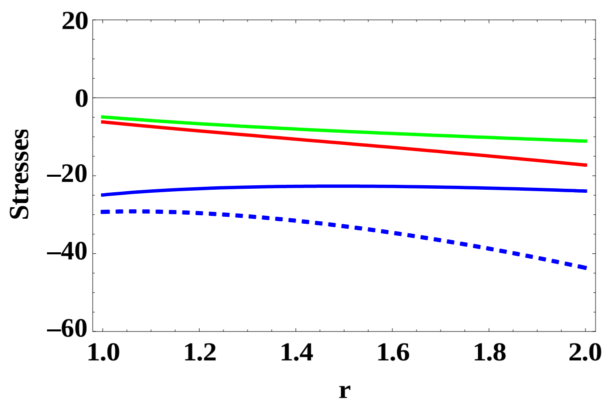

Figure 3. Radial and circumferential stresses for various radii with internal and external pressures as $p_1 = 5$ and $p_2 = 10$, and $k = 1$ and 2, and $C_0 = 0.5$.

Tables 1, 2 and 3 are formulated for radial and circumferential stresses for $k = 1$ and $k = 2$ with different values of C_0. These tables show that the stresses are compressive and increase with the non-homogeneity parameters $k = 1$ and $k = 2$. The positive values of bending moment for $k = 1$ and $k = 2$ shows that stresses are compressive in nature which can also be observed by graphs and tables.

Table 1. Radial and circumferential stresses for various radii with internal and external pressures as $p_1 = 5$ and $p_2 = 10$, and $k = 1$ and 2, and $C_0 = 0.3$.

	$k = 1$		$k = 2$	
r	T_{rr}	$T_{\theta\theta}$	T_{rr}	$T_{\theta\theta}$
1	-5.011	-23.4	-5.56	-25.87
1.2	-6.57	-21.37	-7.35	-24.94
1.4	-7.74	-20.36	-8.85	-25.33
1.6	-8.6	-19.83	-10.2	-26.56
1.8	-9.47	-19.65	-11.51	-28.44
2	-10.16	-19.68	-12.8	-30.85

Table 2. Radial and circumferential stresses for various radii with internal and external pressures as $p_1 = 5$ and $p_2 = 10$, and $k = 1$ and 2, and $C_0 = 0.4$.

	$k = 1$		$k = 2$	
r	T_{rr}	$T_{\theta\theta}$	T_{rr}	$T_{\theta\theta}$
1	-4.95	-24.14	-5.8	-27.53
1.2	-6.5	-22.3	-7.8	-27.21
1.4	-7.8	-21.46	-9.6	-28.33
1.6	-8.9	-21.18	-11.35	-30.47
1.8	-9.78	-21.26	-13.06	-33.39
2	-10.58	-21.58	-14.82	-37.06

Table 3. Radial and circumferential stresses for various radii with internal and external pressures as $p_1 = 5$ and $p_2 = 10$, and $k = 1$ and 2, and $C_0 = 0.5$.

	$k = 1$		$k = 2$	
r	T_{rr}	$T_{\theta\theta}$	T_{rr}	$T_{\theta\theta}$
1	-4.9	-24.94	-6.18	-29.27
1.2	-6.64	-23.31	-8.49	-29.60
1.4	-8.01	-22.72	-10.62	-31.54
1.6	-9.16	-22.74	-12.73	-34.64
1.8	-10.18	-23.18	-14.93	-38.71
2	-11.12	-23.94	-17.26	-43.64

CONCLUSION

Stresses in elastic and plastic state have been evaluated for axi-symmetric FGM plate with bending moment M using the classical theory approach. On the basis of all the calculations and graphs leads to the conclusion that the bending of axi-symmetric wide plates with non-homogeneity parameter ($k = 2$) is safe for designing, as compared to wide plates for $k = 1$, and this is because of the reason that circumferential stresses are lesser for the plate bending with $k = 2$ and $C_0 = 0.5$.

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