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A REVISIT TO THE PROBLEM OF FLOW PAST A PAIR OF SEPARATED SOLID SPHERES

OSVRT NA PROBLEM STRUJANJA PREKO PARA ODVOJENIH ČVRSTIH SFERA

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Abstract

The problem of Stokes flow of a viscous fluid past a pair of separated solid spheres solved by Payne and Pell is revisited in this paper. Payne and Pell worked on the peripolar coordinate system, whereas we consider a bipolar system in this work. One impressive result of this study is that we derived an expression for the drag experienced by the system of two-spheres by modifying the expression that Payne and Pell gave for a general axisymmetric body. Further, this study gave rise to some interesting observations. Though one sphere's presence affects the other, the drag on the system is found equal to the sum of the drag on individual spheres. For spheres of equal radius, we computed the drag on each sphere using the formulae given by Stimson and Jeffery and found that it is precisely half the drag computed on the system. If the spheres are of unequal radius, we arrive at an empirical formula to compute bounds for each sphere's drag. These bounds include values calculated by Jeffery and Stimson in their work on the motion of two spheres in a viscous fluid. We also observe that the drag on the sphere facing the fluid flow first gets saturated at a value that equals the drag on the system with decreasing radius of the other (latter) sphere. Another remarkable feature of our work is that, as a limiting case, we derive the individual spheres' drag, and the values are in excellent agreement with those computed by Stokes formula for drag on a single sphere. Further to these, we have also carried out numerical evaluations for flow visualization and plots of pressure.

Ključne reči
- par odvojenih sfera
- bipolarne koordinate
- Gegenbauer funkcije
- funkcija protoka
- otpor

Izvod

U ovom radu je dat osvrt na problem Stoksovog strujanja viskoznog fluida preko para odvojenih čvrstih sfera, a koji su rešili Pejn i Pel. Pejn i Pel su koristili peripolarni koordinatni sistem, dok u našem radu koristimo bipolarni sistem. Jedan od zanimljivih rezultata u našem radu jeste izvođenje izraza za otpor sistema od dve sfere, modifikovanjem izraza koji su Pejn i Pel dali za opšte osnosimetrično telo. Osim toga, u ovom radu su evidentna interesantna uočavanja. Iako prisustvo jedne sfere utiče na drugu, otpor u sistemu je jednak sumi otpora pojedinačnih sfera. Kod sfera istih poluprečnika, izračunali smo otpor za svaku od sfera koristeći formule Stimsona i Džefrija, pa smo otkrili da je jednak praktično polovini otpora izračunatog za ceo sistem. Ako su sfere različitog poluprečnika, dobijamo empirijski obrazac za izračunavanje veza kod otpora za svaku od sfera. Ove veze sadrže vrednosti koje su izračunali Džefri i Stimson u svom radu o kretanju dve sfere u viskoznom fluidu. Takođe uočavamo da se otpor na sferi, na koju prvo dolazi tok fluida, prvi dostiže kritičnu vrednost koja je jednaka otporu sistema sa smanjenjem poluprečnika druge sfere. Druga značajna karakteristika u našem radu je, u graničnom slučaju, dolazimo do rešenja za otpor za pojedinačne sfere, a dobijene vrednosti se izvanredno slažu sa vrednostima dobijenim prema formuli Stoksa za otpor pojedinačne sfere. Daljim numeričkim proračunom dobijamo rešenja za vizualizaciju strujanja i dijagrame pritiska.

INTRODUCTION

The present work aims to revisit the problem of Stokes' flow of viscous fluid past a pair of separated solid spheres solved by Payne and Pell, /1/. The formula for drag derived by them in terms of stream function is modified to calculate the drag experienced by the two-spheres system. It is for the reference of the readers that, in their work, Payne and Pell worked on the problem of viscous fluid flow past a pair of separated spheres in the peripolar coordinate system and determined stream function in terms of associated Legendre's polynomials. They also derived an expression for determining the drag experienced by a general axisymmetric body in terms of the stream function. However, they have not applied the same to derive the drag for separated spheres. Instead, they referred to Stimson and Jeffery, /2/, who solved an equivalent problem in the bipolar system and cited the expression from their earlier work for the drag on each sphere when the two spheres are of equal radius. It is to be noted that Stimson and Jeffery derived the expression for drag using the conventional formula that is in terms of the axial component of the stress tensor. Further, the expression for pressure experienced by the spheres is not mentioned explicitly in both of these papers.

Considering the above observations, we revisited the two-sphere problem. As mentioned earlier, we formulated the problem in the bipolar coordinate system, and we computed an analytical expression for the stream and pressure functions and derived the drag experienced by the system of spheres by extending the formula given by Payne and Pell, /1/. Further, to depict the flow profile, numerical evaluation of stream function expression is carried out by developing MATHEMATICA® codes. We then presented the flow profiles and pressure distribution by varying the radii of the spheres. A notable aspect of this study is that our approach to compute drag using stream function gave the drag experienced by the system of spheres instead of an individual sphere. Furthermore, we derived empirical formulae for bounds for the drag on each sphere and compared them with those published in the literature, /2/. As a limiting case, we also computed the drag experienced by individual (single) spheres. The values obtained are in good agreement with those computed using the well-known Stokes formula for drag on a single sphere.

MATHEMATICAL FORMULATION OF THE PROBLEM

Consider the flow of a viscous fluid (with a uniform velocity U at infinity in negative x-direction) past a pair of separated solid spheres that are fixed in the flow domain, as shown in Fig. 1.

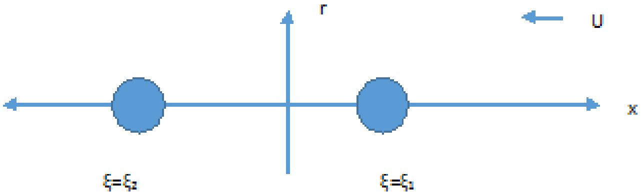

Figure 1. Schematic diagram of flow past separated spheres in bipolar coordinates.

Corresponding scale factors are taken to describe the flow domain, where

$$x=\frac{a\sinh\xi}{\cosh\xi-\cos\eta} \quad \text{and} \quad r=\frac{a\sin\eta}{\cosh\xi-\cos\eta},$$

$$h_\xi=\frac{a}{\cosh\xi-\cos\eta}, \quad h_\eta=\frac{a}{\cosh\xi-\cos\eta},$$

$$h_\varphi=\frac{a\sin\eta}{\cosh\xi-\cos\eta}, \tag{2}$$

with $-\infty<\xi<\infty$, $0<\eta<\pi$.

In this coordinate system, the equation $\xi=c>0$ (where c is constant) represents a sphere with its centre on the positive x-axis placed at a distance a coth$c$ from the origin (along the x-axis), and radius equals a cosech$c$. Whereas $\xi=c<0$ describes a sphere on the negative x-axis with its centre at a distance $-a$ coth$c$ from the origin (along the x-axis) with radius of a cosech$c$.

Assuming that fluid flow is axisymmetric, the velocity vector takes the form $\vec{q}=u(\xi,\eta)\hat{e}_\xi+v(\xi,\eta)\hat{e}_\eta$, and the pressure is $p(\xi,n)$. Further, considering the fluid to be incompressible and the flow as steady, Navier-Stokes momentum equations under Stokesian approximation take the form /3/:

$$\operatorname{grad} p+\mu\operatorname{curl}\operatorname{curl}(\vec{q})=0\,. \tag{3}$$

Now, introducing the stream function through

$$h_\eta h_\varphi u=-\frac{\partial\psi}{\partial\eta}; \quad h_\xi h_\varphi u=\frac{\partial\psi}{\partial\xi}, \tag{4}$$

we see that

$$\operatorname{curl}\vec{q}=\left(\frac{1}{h_\varphi}E^2\psi\right)\hat{e}_\varphi, \tag{5}$$

$$\operatorname{curl}\operatorname{curl}\vec{q}=\frac{1}{h_\xi h_\eta h_\varphi}\left(h_\xi\frac{\partial}{\partial\eta}(E^2\psi)\hat{e}_\xi-h_\eta\frac{\partial}{\partial\xi}(E^2\psi)\hat{e}_\eta\right), \tag{6}$$

in which the Stokes stream function operator E^2 is given by

$$E^2=\frac{h_\varphi}{h_\xi h_\eta}\left\{\frac{\partial}{\partial\xi}\left(\frac{h_\eta}{h_\xi h_\varphi}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_\xi}{h_\eta h_\varphi}\frac{\partial}{\partial\eta}\right)\right\}. \tag{7}$$

Using Eqs.(5) and (6), Eq.(3) takes the form

$$\frac{1}{h_\xi}\frac{\partial p}{\partial\xi}+\frac{\mu}{h_\eta h_\varphi}\frac{\partial}{\partial\eta}(E^2\psi)=0, \tag{8}$$

$$\frac{1}{h_\eta}\frac{\partial p}{\partial\eta}-\frac{\mu}{h_\xi h_\varphi}\frac{\partial}{\partial\xi}(E^2\psi)=0. \tag{9}$$

Eliminating p from Eq.(8) and Eq.(9) gives

$$E^4\psi=0, \tag{10}$$

which is the equation governing the fluid flow in the problem considered.

The determination of the relevant flow field variables ψ and p is subjected to the following boundary and regularity conditions.

No-slip boundary condition, i.e.

$$\psi=0 \quad \text{and} \quad \frac{\partial\psi}{\partial\xi}=0 \quad \text{on} \quad \xi=\xi_1 \quad \text{and} \quad \xi=\xi_2. \tag{11}$$

Velocities are regular on the axis, and far away from the spheres the flow is a uniform stream which means, at infinity

$$\psi=-\frac{1}{2}Ur^2. \tag{12}$$

(1) *Solution to the Eq.(10)*

In view of the linearity of Eq.(10), we assume its solution in the form

$$\psi=\psi_0+\psi_1, \tag{13}$$

where the function ψ_0 in Eq.(13) represents the stream function due to a uniform stream of magnitude U parallel to the axis of symmetry, far away from the spheres.

Thus, using Eq.(12), we get

$$\psi_0=-\frac{1}{2}U\left(\frac{a\sin\eta}{\cosh\xi-\cos\eta}\right)^2. \tag{14}$$

It can be easily verified that $E^2\psi_0=0$, and hence, $E^4\psi_0=0$.

Now, ψ_1 has to be found such that it satisfies Eq.(10) together with conditions given in Eq.(11).

For this, we consider Eq.(10) and re-write it as

$$E^4\psi=E^4(\psi_0+\psi_1)=E^4\psi_1=E^2(E^2\psi_1)=0 \tag{15}$$

(from the reason mentioned above).

Let
$$E^2\psi_1=f, \tag{16}$$

then Eq.(15) can be written as

$$E^2 f = 0 . \tag{17}$$

We shall now solve Eq.(17) for f and then substitute it in Eq.(16) and again solve it for ψ_1.

Solution to Eq.(17)

Using the expressions in Eqs.(1) and (2), Eq.(17) in the bipolar coordinate system takes the form,

$$\frac{\cosh\xi-\cos\eta}{a^2}\left[a\sin\eta\frac{\partial}{\partial\xi}\left(\frac{\cosh\xi-\cos\eta}{a\sin\eta}\frac{\partial}{\partial\xi}\right)+\right.$$

$$\left.+\frac{\partial}{\partial\eta}\left((\cosh\xi-\cos\eta)\frac{\partial}{\partial\eta}\right)\right]f=0 . \tag{18}$$

Following /2/, let us denote $\cos\eta = \tau$. Then Eq.(18) takes the form

$$\frac{\cosh\xi-\tau}{a^2}\left[\frac{\partial}{\partial\xi}\left((\cosh\xi-\tau)\frac{\partial}{\partial\xi}\right)+\right.$$

$$\left.+(1-\tau^2)\frac{\partial}{\partial\tau}\left((\cosh\xi-\tau)\frac{\partial}{\partial\tau}\right)\right]f=0 . \tag{19}$$

We shall use the method of separation of variables to find the solution to Eq.(19). For this, let us assume its solution as

$$f(\xi,\tau)=(\cosh\xi-\tau)^n g(\xi,\tau) . \tag{20}$$

Substituting the expression in Eq.(20) in Eq.(19) and after a straightforward calculation, we get

$$n^2(\cosh\xi-\tau)g+(n\cosh\xi)g+(2n+1)\left(\sinh\xi\frac{\partial g}{\partial\xi}-(1-\tau^2)\frac{\partial g}{\partial\tau}\right)+$$

$$+(\cosh\xi-\tau)\left(\frac{\partial^2 g}{\partial\xi^2}+(1-\tau^2)\frac{\partial^2 g}{\partial\tau^2}\right)=0 . \tag{21}$$

It can be clearly seen that $n = -1/2$ reduces Eq.(21) into a variable separable form. Thus, choosing $n = -1/2$, Eq.(21) reduces to

$$-\frac{(\cosh\xi-\tau)}{4}g+(\cosh\xi-\tau)\left(\frac{\partial^2 g}{\partial\xi^2}+(1-\tau^2)\frac{\partial^2 g}{\partial\tau^2}\right)=0 . \tag{22}$$

We further take

$$g(\xi,\tau)=X(\xi)Y(\tau) . \tag{23}$$

Then, Eq.(22) takes the form

$$4\frac{X''}{X}=-4\frac{(1-\mu^2)Y''}{Y}+1=\lambda . \tag{24}$$

This equation has a non-trivial solution for $(\lambda-1)/4 = n(n + 1)$.

Thus, for this choice of λ, we get $Y(\tau) = \vartheta_{n+1}(\tau)$, which is the Gegenbaur function of degree -1/2 and of I kind /3/, $\vartheta_{n+1}(\tau)=\dfrac{P_{n-1}(\tau)-P_{n+1}(\tau)}{2n+1}$, $n \geq 1$, where $P_n(\tau)$ are the Legendre polynomials. Also,

$$X(\xi)=c_1\cosh\left(n+\frac{1}{2}\right)\xi+c_2\sinh\left(n+\frac{1}{2}\right)\xi . \tag{25}$$

Thus, the solution of Eq.(18) is

$$f(\xi,\tau)=(\cosh\xi-\tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+\right.$$

$$\left.+\left(B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau)\right), \tag{26}$$

where: A_n, B_n are arbitrary constants.

Solution to Eq.(16)

Using the Eq.(26), Eq.(16) takes the form,

$$\frac{\cosh\xi-\tau}{a^2}\left(\frac{\partial}{\partial\xi}\left[(\cosh\xi-\tau)\frac{\partial}{\partial\xi}\right]+(1-\tau^2)\frac{\partial}{\partial\tau}\left[(\cosh\xi-\tau)\frac{\partial}{\partial\tau}\right]\right)\psi_1=$$

$$=(\cosh\xi-\tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau) \tag{27}$$

Again, using the method of separation of variables, the solution of Eq.(27) is taken in the form

$$\psi_1(\xi,\tau)=a^2(\cosh\xi-\tau)^{-3/2}\sum_{n=1}^{\infty}H_n(\xi)\vartheta_{n+1}(\tau) . \tag{28}$$

Substituting the above expression in Eq.(27), we now have to derive the equation governing $H_n(\xi)$. For this, we use the following relations in Gegenbaur functions as mentioned below, /4/:

$$(1-x^2)\vartheta'_{n+1}(x)=-(n+1)x\vartheta_{n+1}(x)+(n-1)\vartheta_n(x) , \tag{29}$$

$$(n+1)\vartheta_{n+1}(x)=(2n-1)x\vartheta_n(x)-(n-2)\vartheta_{n-1}(x) . \tag{30}$$

Further, on applying the orthogonality property of Legendre polynomials, the equation governing $H_n(x)$ is given as

$$(\cosh\xi)H''_n-(2\sinh\xi)H'_n+\left(-n(n+1)+\frac{3}{4}\right)(\cosh\xi)H_n=$$

$$=A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi . \tag{31}$$

Now, in view of the form of Eq.(31), assume its solution as

$$H_n=C\cosh\left(n+\frac{1}{2}\right)\xi+\sinh\left(n+\frac{1}{2}\right)\xi . \tag{32}$$

Substituting the expression of H_n in Eq.(31) yields,

$$H_n=C_n\cosh\left(n-\frac{1}{2}\right)\xi+D_n\sinh\left(n-\frac{1}{2}\right)\xi+$$

$$+E_n\cosh\left(n+\frac{3}{2}\right)\xi+F_n\sinh\left(n+\frac{3}{2}\right)\xi . \tag{33}$$

Thus, we have

$$\psi_1(\xi,\tau)=a^2(\cosh\xi-\tau)^{-3/2}\sum_{n=1}^{\infty}\left(C_n\cosh\left(n-\frac{1}{2}\right)\xi+D_n\times\right.$$

$$\left.\times\sinh\left(n-\frac{1}{2}\right)\xi+E\cosh\left(n+\frac{3}{2}\right)\xi+F_n\sinh\left(n+\frac{3}{2}\right)\xi\right)\vartheta_{n+1}(\tau) \tag{34}$$

where: $-(2n-1)C_n+(2n+3)E_n=A_n$,

and $-(2n-1)D_n+(2n+3)F_n=B_n$. (35)

Thus, the solution to Eq.(10) is

$$\psi(\xi,\tau)=-\frac{1}{2}U\left(\frac{a\sqrt{1-\tau^2}}{\cosh\xi-\tau}\right)^2+a^2(\cosh\xi-\tau)^{-3/2}\times$$

$$\times\sum_{n=1}^{\infty}\left(C_n\cosh\left(n-\frac{1}{2}\right)\xi+D_n\sinh\left(n-\frac{1}{2}\right)\xi+E_n\cosh\left(n+\frac{3}{2}\right)\xi+\right.$$

$$+F_n \sinh\left(n+\frac{3}{2}\right)\xi\Bigg)\vartheta_{n+1}(\tau)\,. \qquad (36)$$

DETERMINATION OF ARBITRARY CONSTANTS

The four sets of arbitrary constants in Eq.(36) are determined using the boundary conditions given in Eq.(10) as follows:

(i) $\psi=0$ at $\xi=\xi_1$ gives

$$\sum_{n+1}^{\infty}\Bigg(C_n\cosh\left(n-\frac{1}{2}\right)\xi_1+D_n\sinh\left(n-\frac{1}{2}\right)\xi_1+E_n\cosh\left(n+\frac{3}{2}\right)\xi_1+$$
$$+F_n\sinh\left(n+\frac{3}{2}\right)\xi_1\Bigg)\vartheta_{n+1}(\tau)=\frac{1}{2}U(1-\tau^2)(\cosh\xi_1-\tau)^{-1/2}\,, \quad (37)$$

(ii) $\psi=0$ at $\xi=\xi_2$ gives

$$\sum_{n+1}^{\infty}\Bigg(C_n\cosh\left(n-\frac{1}{2}\right)\xi_2+D_n\sinh\left(n-\frac{1}{2}\right)\xi_2+E_n\cosh\left(n+\frac{3}{2}\right)\xi_2+$$
$$+F_n\sinh\left(n+\frac{3}{2}\right)\xi_2\Bigg)\vartheta_{n+1}(\tau)=\frac{1}{2}U(1-\tau^2)(\cosh\xi_2-\tau)^{-1/2}\,, \quad (38)$$

(iii) $\dfrac{\partial\psi}{\partial\xi}=0$ at $\xi=\xi_1$ gives

$$\sum_{n=1}^{\infty}\Bigg(\left(n-\frac{1}{2}\right)C_n\sinh\left(n-\frac{1}{2}\right)\xi_1+\left(n-\frac{1}{2}\right)D_n\cosh\left(n-\frac{1}{2}\right)\xi_1+$$
$$+\left(n+\frac{3}{2}\right)E_n\sinh\left(n+\frac{3}{2}\right)\xi_1+\left(n+\frac{3}{2}\right)F_n\cosh\left(n+\frac{3}{2}\right)\xi_1\Bigg)\vartheta_{n+1}(\tau)=$$
$$=-\frac{1}{4}U(1-\tau^2)\sinh\xi_1(\cosh\xi_1-\tau)^{-3/2}\,, \qquad (39)$$

(iv) $\dfrac{\partial\psi}{\partial\xi}=0$ at $\xi=\xi_2$ gives

$$\sum_{n=1}^{\infty}\Bigg(\left(n-\frac{1}{2}\right)C_n\sinh\left(n-\frac{1}{2}\right)\xi_2+\left(n-\frac{1}{2}\right)D_n\cosh\left(n-\frac{1}{2}\right)\xi_2+$$
$$\left(n+\frac{3}{2}\right)E_n\sinh\left(n+\frac{3}{2}\right)\xi_2+\left(n+\frac{3}{2}\right)F_n\cosh\left(n+\frac{3}{2}\right)\xi_2\Bigg)\vartheta_{n+1}(\tau)=$$
$$=-\frac{1}{4}U(1-\tau^2)\sinh\xi_2(\cosh\xi_2-\tau)^{-3/2}\,. \qquad (40)$$

Now, to find the four sets of constants, we consider the relations from /5/ as shown below:

$$\int_{-1}^{1}\frac{P_n(x)}{(\cosh\xi-x)^{1/2}}dx=\frac{2\sqrt{2}}{2n+1}e^{-\left(n+\frac{1}{2}\right)|\xi|}\,, \qquad (41)$$

$$\int_{-1}^{1}\frac{P_n(x)}{(\cosh\xi-x)^{3/2}}dx=\frac{2\sqrt{2}}{\sinh|\xi|}e^{-\left(n+\frac{1}{2}\right)|\xi|}\,. \qquad (42)$$

The orthogonality relation of Gegenbauer functions from /4/,

$$\int_{-1}^{1}\frac{\vartheta_m(x)\vartheta_n(x)}{1-x^2}dx=\begin{cases}0, & m\neq n\\ \dfrac{2}{n(n-1)(2n-1)}, & m=n\end{cases} \qquad (43)$$

Now, using relations Eq.(41) to Eq.(43), we see that Eqs.(37)-(40) take the form

$$C_n\cosh\left(n-\frac{1}{2}\right)\xi_1+D_n\sinh\left(n-\frac{1}{2}\right)\xi_1+E_n\cosh\left(n+\frac{3}{2}\right)\xi_1+F_n\times$$
$$\times\sinh\left(n+\frac{3}{2}\right)\xi_1=k\left((2n+3)e^{-\left(n-\frac{1}{2}\right)\xi_1}-(2n-1)e^{-\left(n+\frac{3}{2}\right)\xi_1}\right), \quad (44)$$

$$C_n\cosh\left(n-\frac{1}{2}\right)\xi_2+D_n\sinh\left(n-\frac{1}{2}\right)\xi_2+E_n\cosh\left(n+\frac{3}{2}\right)\xi_2+F_n\times$$
$$\times\sinh\left(n+\frac{3}{2}\right)\xi_2=k\left((2n+3)e^{-\left(n-\frac{1}{2}\right)\xi_2}-(2n-1)e^{-\left(n+\frac{3}{2}\right)\xi_2}\right), \quad (45)$$

$$\left(n-\frac{1}{2}\right)C_n\cosh\left(n-\frac{1}{2}\right)\xi_1+\left(n-\frac{1}{2}\right)D_n\sinh\left(n-\frac{1}{2}\right)\xi_1+$$
$$+\left(n+\frac{3}{2}\right)E_n\cosh\left(n+\frac{3}{2}\right)\xi_1+\left(n+\frac{3}{2}\right)F_n\sinh\left(n+\frac{3}{2}\right)\xi_1=$$
$$=-\frac{(2n-1)(2n+3)k}{2}\left(e^{-\left(n-\frac{1}{2}\right)\xi_1}-e^{-\left(n+\frac{3}{2}\right)\xi_1}\right), \quad (46)$$

$$\left(n-\frac{1}{2}\right)C_n\cosh\left(n-\frac{1}{2}\right)\xi_2+\left(n-\frac{1}{2}\right)D_n\sinh\left(n-\frac{1}{2}\right)\xi_2+$$
$$+\left(n+\frac{3}{2}\right)E_n\cosh\left(n+\frac{3}{2}\right)\xi_2+\left(n+\frac{3}{2}\right)F_n\sinh\left(n+\frac{3}{2}\right)\xi_2=$$
$$=-\frac{(2n-1)(2n+3)k}{2}\left(e^{-\left(n-\frac{1}{2}\right)\xi_2}-e^{-\left(n+\frac{3}{2}\right)\xi_2}\right), \quad (47)$$

where
$$k=U\frac{n(n+1)}{\sqrt{2}(2n-1)(2n+3)}\,. \qquad (48)$$

Solving Eqs.(45) to (48) gives expressions for C_n, D_n, E_n, F_n, and hence the stream function is entirely determined.

DETERMINATION OF DRAG

To determine the spheres' drag, we use the formula given by Payne and Pell for a general axisymmetric body, /1/. For the present problem, it takes the form

$$F_z=-8\pi\mu\lim_{\substack{\xi\to 0\\ \eta\to 2\pi}}\frac{\rho(\psi-\psi_\infty)}{r^2}(=) \qquad (49)$$

$$=-8\pi\mu\lim_{\substack{\xi\to 0\\ \eta\to 2\pi}}\frac{\rho(\cosh\xi-\tau)^{1/2}}{(1-\tau^2)}\sum_{n=1}^{\infty}\Bigg(C_n\cosh\left(n-\frac{1}{2}\right)\xi+D_n\times$$
$$\times\sinh\left(n-\frac{1}{2}\right)\xi+E_n\cosh\left(n+\frac{3}{2}\right)\xi+F_n\sinh\left(n+\frac{3}{2}\right)\xi\Bigg)\vartheta_{n+1}(\tau) \quad (50)$$

(derived using Eq.(36)).

Here,

$$\rho=\sqrt{r^2+x^2}=a\frac{\sqrt{\sin^2\eta+\sinh^2\xi}}{\cosh\xi-\cos\eta} \quad \text{(from Eq.(1)).} \qquad (51)$$

It is to be noted that this formula gives us the total drag experienced by the system of separated spheres instead of the drag on each sphere. We then compute the drag experienced by each sphere if the two spheres are of equal radius and derive bounds for each of the spheres' drag experienced when the spheres are of an unequal radius in Case (I) and Case (II) presented below.

Case (I): Equal spheres. $|\xi_1| = |\xi_2| = \alpha$ (say).

Solving Eqs.(44)-(47), we get
$D_n = 0$, $F_n = 0$ and

$$\begin{bmatrix} C_n \\ E_n \end{bmatrix} = \begin{bmatrix} \sinh\left(n-\frac{1}{2}\right)\alpha & \sinh\left(n+\frac{1}{2}\right)\alpha \\ \left(n-\frac{1}{2}\right)\sinh\left(n-\frac{1}{2}\right)\alpha & \left(n+\frac{3}{2}\right)\sinh\left(n+\frac{3}{2}\right)\alpha \end{bmatrix}^{-1} \times$$
$$\times \begin{bmatrix} k\left((2n+3)e^{-\left(n-\frac{1}{2}\right)\alpha} - (2n-1)e^{-\left(n+\frac{3}{2}\right)\alpha}\right) \\ -(2n-1)(2n+3)\dfrac{k}{2}\left(e^{-\left(n-\frac{1}{2}\right)\alpha} - e^{-\left(n+\frac{3}{2}\right)\alpha}\right) \end{bmatrix}. \quad (52)$$

Now, using Eqs.(50) and (52), the drag on the system of spheres simplifies to

$$F_z = -8\pi\mu aU \lim_{\substack{\xi\to 0 \\ \eta\to 2\pi}} \frac{\rho(\cosh\xi-\tau)^{1/2}}{(1-\tau^2)} \sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{2}(2n-1)(2n+3)} \times$$
$$\times \left(\frac{-(2n+3)\left(e^{2\alpha}(2n+1) - 2n+1 - 2e^{-(2n+1)\alpha}\right) +}{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha} \right.$$
$$\left. \frac{+(2n-1)\left((1-e^{-2\alpha})(2n+1) + 2\left(1-e^{-(2n+1)\alpha}\right)\right)}{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha} \right) \vartheta_{n+1}(\tau) \quad (53)$$

and the non-dimensional drag is taken as

$$D = -8\pi \lim_{\substack{\xi\to 0 \\ \eta\to 2\pi}} \frac{\rho(\cosh\xi-\tau)^{1/2}}{(1-\tau^2)} \sum_{n=1}^{\infty} \frac{n(n+1)}{\sqrt{2}(2n-1)(2n+3)} \times$$
$$\times \left(\frac{-(2n+3)\left(e^{2\alpha}(2n+1) - 2n+1 - 2e^{-(2n+1)\alpha}\right) + (2n-1)\times}{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha} \right.$$
$$\left. \frac{\times\left(\left(1-e^{-2\alpha}\right)(2n+1) + 2\left(1-e^{-(2n+1)\alpha}\right)\right)}{2\sinh(2n+1)\alpha + (2n+1)\sinh 2\alpha} \right) \vartheta_{n+1}(\tau), \quad (54)$$

where: $D = F_z/\mu aU$.

To understand the effect of α on drag, we evaluate the Eq.(54) using MATHEMATICA® and present the non-dimensional drag experienced by each sphere in Table 1, shown below. For the reader's reference, we also compute the non-dimensional radius $r^* = r/a = \operatorname{cosec} h(\alpha)$ and the centre's distance from the origin $d^* = d/a = \coth(\alpha)$ for each α.

Considering Table 1, we infer from columns 4 and 5 that, in the case of equal spheres, the drag on the system of two spheres equals twice the drag on each sphere (computed using the formula on drag given in /1/ and /2/). Thus, knowing the drag experienced by the system, we can determine the drag on individual spheres.

Table 1. Values of non-dimensional drag for different α.

α	Radius (non-dimensional) r^*	Distance of the centre of each sphere from the origin (non-dimensional) d	Drag on the system D	Drag on each sphere computed using the formula given in /2/
0.05	19.9917	20.0167	73.735	36.8675
0.1	9.983	10.033	73.1885	36.5943
0.5	1.919	2.164	45.5071	22.7536
1.0	0.8509	1.313	22.524	11.262
2.0	0.2757	1.037	8.6918	4.3459
5.0	0.0135	1.000	0.50297	0.2515

A plot of the variation of drag vs α is as shown in Fig. 2. This plot shows that each sphere's drag experienced in the two-sphere problem is not a linear function of the sphere's radius, unlike the drag in the case of a single sphere. (It is to be noted that the drag on a single sphere is computed using the non-dimensional form of the Stokes formula given by $F = 6\pi r^*$. We also observe that in the two-sphere problem, the drag experienced by each sphere remains constant beyond a specific value of its radius, indicating that the presence of one sphere limits the drag on the other.

We know from Stokes's formula that the drag experienced by a sphere of radius r, fixed in the flow domain of a viscous fluid of viscosity μ, streaming with a uniform velocity $-U$ at infinity is $F = -6\pi\mu Ur$. Let us define λ as the ratio of the drag experienced by either sphere in the other's presence to the drag computed using Stokes's formula.

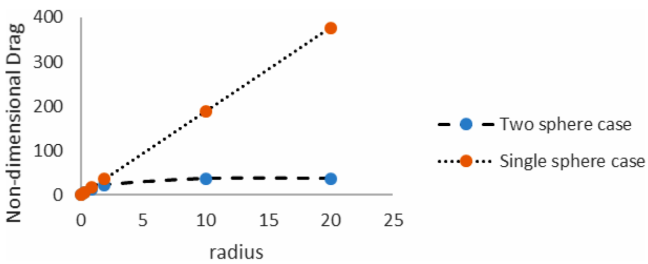

Figure 2. Variation of non-dimensional drag with sphere radius.

For each α, we present the value of λ in Table 2. We see that these values match those mentioned by Jeffery and Stimson in their work (equivalent to the present work) on finding the force necessary to maintain the sphere's motion in a viscous fluid, /2/.

Table 2. Values of λ for a given α.

α	λ
0.05	0.1811
0.1	0.3479
0.5	0.6591
1.0	0.7024
2.0	0.8362
5.0	0.9899

Case (II): Two spheres are of an unequal radius.

As discussed earlier, the expression for drag given in formulae in Eq.(50) gives the total drag experienced by the two-spheres system. We compute the value of non-dimensional drag for specific values of ξ_1, ξ_2 and the same is shown in column 3 of Table 3.

To validate the formula, we compute the drag experienced by each of the spheres using the formulae given in /2/. It is seen here again (as in Case I) that the sum of the drags presented in column 4 of Table 3 matches with the drag of the system (column 3).

Table 3. Non-dimensional drag for different pairs (ξ_1, ξ_2).

ξ_1	ξ_2	Drag experienced by system of separated spheres (D)	Drag computed using formulae in /2/ $\xi=\xi_1$	$\xi=\xi_2$
0.5	-1.0	38.2989	31.7176	6.59709
1.5	-1.0	18.7752	5.2842	13.491
2.0	-1.0	17.4197	2.80943	14.6102
1.0	-0.5	38.2989	6.59709	31.7176
1.0	-1.5	18.7752	13.491	5.2842
1.0	-2.0	17.4197	14.6102	2.80943

Now, we proceed to identify the bounds for the drag (non-dimensional) experienced by each of the spheres.

As seen from Stokes's formula for drag, we understand that the drag experienced by a sphere is proportional to its radius. Thus, if D_1, D_2 and D are the drags experienced by spheres $\xi=\xi_1$, $\xi=\xi_2$, and the system of spheres, respectively, then we have

$$\frac{D_1}{\cosh\xi_1}=\frac{D_2}{\cosh|\xi_2|}=\frac{D_1+D_2=D}{\cosh\xi_1+\cosh|\xi_2|}\,. \tag{55}$$

Since, in the two-sphere problem, the presence of one sphere affects the other, we get only the bounds for the drag experienced by each of them. Thus, we have

$$D_1\begin{cases}\geq\dfrac{D\,\mathrm{cosec}\,h(\xi_1)}{\mathrm{cosec}\,h(\xi_1)+\mathrm{cosec}\,h(|\xi_2|)}, & \xi_1\leq|\xi_2|\\[2ex] <\dfrac{D\,\mathrm{cosec}\,h(\xi_1)}{\mathrm{cosec}\,h(\xi_1)+\mathrm{cosec}\,h(|\xi_2|)}, & \xi_1>|\xi_2|\end{cases}$$

Further, for larger values of $|\xi_2|$, $D_1\approx D$. (56)

$$\text{Similarly,}\quad D_2\begin{cases}\geq\dfrac{D\,\mathrm{cosec}\,h(|\xi_2|)}{\mathrm{cosec}\,h(\xi_1)+\mathrm{cosec}\,h(|\xi_2|)}, & |\xi_2|\leq\xi_1\\[2ex] <\dfrac{D\,\mathrm{cosec}\,h(|\xi_2|)}{\mathrm{cosec}\,h(\xi_1)+\mathrm{cosec}\,h(|\xi_2|)}, & |\xi_2|>\xi_1\end{cases}$$

and for larger values of ξ_1, $D_2\approx 0$. (57)

We plot (Figs. 3 and 4) the drag on each sphere using the formula in /2/ and the bounds for the drag, using the formulae given above. These plots show that the drag value lies well within the bounds (limits) derived in this work.

Special case

The formula for drag given in Eq.(50) can give the drag experienced by a single sphere in the limiting case as $\xi_1\to\infty$ or $\xi_2\to-\infty$. Table 4 presents the values of non-dimensional drag computed using Eq.(50) for a given ξ_1 with $\xi_2\to-\infty$ (in column 2), and drag computed using Stokes' formula (in column 3). These values agree with each other. It is found that by fixing $\xi_2=-\xi_1$ and letting $\xi_1\to\infty$, the drag computed is the same as the ones shown below.

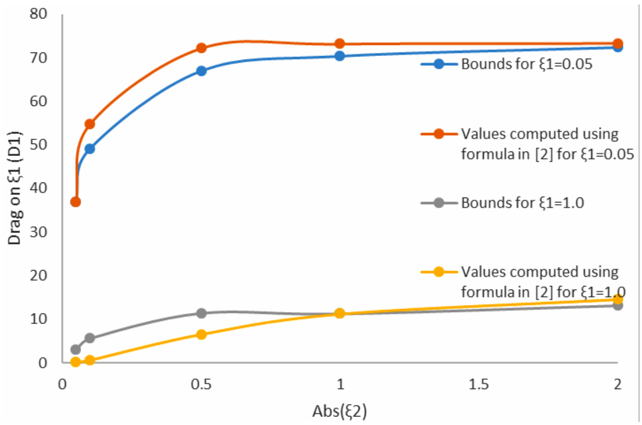

Figure 3. Plot of the drag on $\xi=\xi_1$.

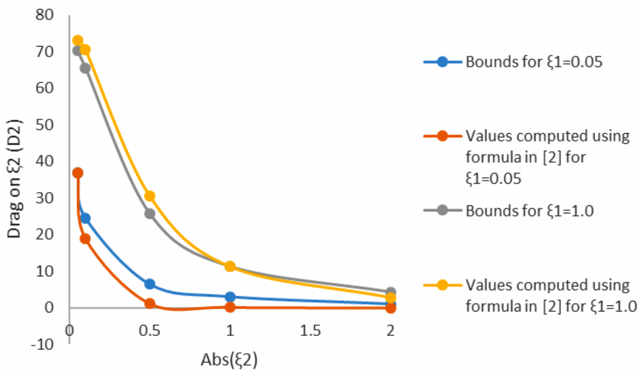

Figure 4. Plot of the drag on $\xi=\xi_2$.

Table 4. Comparison of non-dimensional drag on a single sphere using Eq.(50) and Stokes formula.

ξ_1	Drag computed using Eq.(50)	Drag computed using Stokes formula
0.5	36.173	36.173
1.5	8.8525	8.8525
2.0	5.1972	5.1972

Expression for the pressure

We now derive the expression for the pressure function from Eqs.(8) and (9). For this, let us consider Eqs.(8) and (9) and substitute the expressions for the scale factors from Eq.(2) to get

$$\frac{\partial p}{\partial\xi}=-\frac{\nu(\cosh\xi-\tau)}{a}\frac{\partial}{\partial\tau}(E^2\psi)\,, \tag{58}$$

$$\frac{\partial p}{\partial\tau}=-\frac{\nu(\cosh\xi-\tau)}{a(1-\tau^2)}\frac{\partial}{\partial\xi}(E^2\psi)\,. \tag{59}$$

Eliminating ψ from these equations, we get

$$\frac{\partial}{\partial\xi}\left((\cosh\xi-\tau)^{-1}\frac{\partial p}{\partial\xi}\right)+\frac{\partial}{\partial\tau}\left((\cosh\xi-\tau)^{-1}(1-\tau^2)\frac{\partial p}{\partial\tau}\right)=0\,. \tag{60}$$

Let us assume its solution as

$$p(\xi,\eta)=(\cosh\xi-\tau)^n h(\xi,\tau)\,. \tag{61}$$

Substituting the expression for pressure in Eq.(61) into Eq.(60), we get

$$(\cosh\xi-\tau)(1-\tau^2)\frac{\partial^2 h}{\partial\tau^2}+(2n-1)\sinh\xi\frac{\partial h}{\partial\xi}+(\cosh\xi-\tau)\frac{\partial^2 h}{\partial\xi^2}+$$
$$+\left(1+\tau^2-2n(1-\tau^2)-2\tau\cosh\xi\right)\frac{\partial h}{\partial\tau}+n\left(n\tau+(n-1)\cosh\xi\right)h=0 \tag{62}$$

Take $n = 1/2$ and $h(\xi, r) = X_1(\xi)Y_1(\tau)$ to reduce Eq.(62) into variable separable form given by

$$\frac{X_1''}{X_1}-\frac{1}{4}=-\frac{(1-\tau^2)Y_1''-2Y_1'}{Y_1}=l\ . \tag{63}$$

Choosing $l = n(n + 1)$, we have

$$p(\xi,\tau)=(\cosh\xi-\tau)^{1/2}\sum_{n=0}^{\infty}\left(H_n\cosh\left(n+\frac{1}{2}\right)\xi+\right.$$
$$\left.+G_n\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau)\,, \tag{64}$$

where: ($'$) are Legendre's polynomials. Now, using Eqs.(58) and (59), the H_n and ($'$) can be written in terms of C_n, D_n, E_n, and F_n, thus the pressure function is entirely determined. (Details are provided in the Appendix.)

We now present the streamlines and pressure distribution in cases where spheres are of equal and unequal radius.

Case (I): The case of equal spheres, $|\xi_1| = |\xi_1| = \alpha$.

Plots for streamlines

We develop a code in MATHEMATICA® to compute the coefficients C_n, D_n, E_n, and F_n, n = 1,2,3, … using the Eqs.(44) to (47) and evaluate the non-dimensional stream function $\psi^* = \psi/Ua^2$ using Eq.(36). For plotting the streamlines, this expression is written in the Cartesian coordinate system using the inverse transformation of expressions shown in Eq.(1).

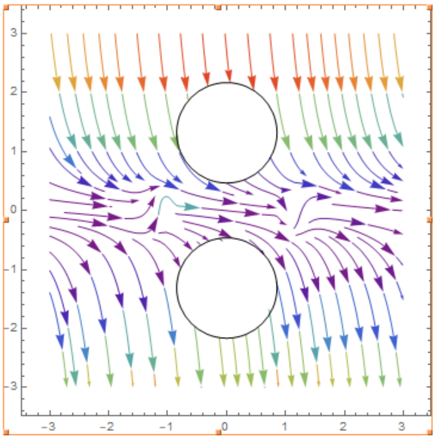

Figure 5. Streamlines in the *xy* plane for $\alpha = 1.0$.

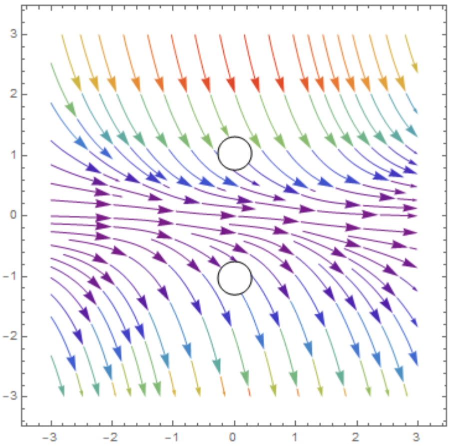

Figure 6. Streamlines in the *xy* plane for $\alpha = 2.0$.

Plots of streamlines for $\alpha = 1.0$ and $\alpha = 2.0$ are presented in Figs. 5 and 6, respectively. Figure 5 depicts the flow near the spheres shows disturbance when spheres are large rather than when they are small (Fig. 6).In Figs. 7 and 8 we present the non-dimensional pressure contours for $\alpha = 1.0$ and $\alpha = 2.0$, respectively.

As mentioned earlier, the disturbance in the flow around larger spheres (Fig. 5) is larger. It is mirrored evidently in the pressure distribution from Figs. 7 and 8. Moreover, the pressure scale is broader in spheres of larger radius than in smaller ones.

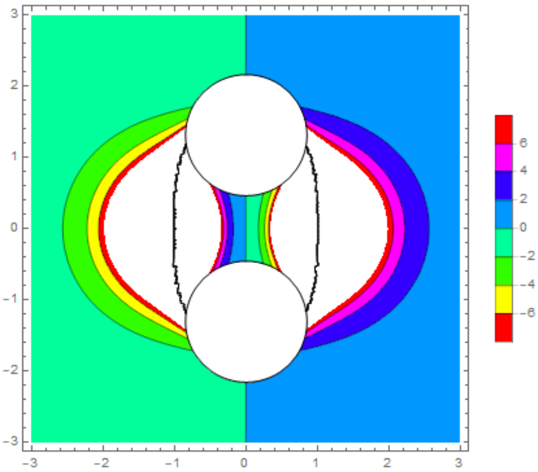

Figure 7. Plot of the non-dimensional pressure function in the *xy* plane for $\alpha = 1.0$.

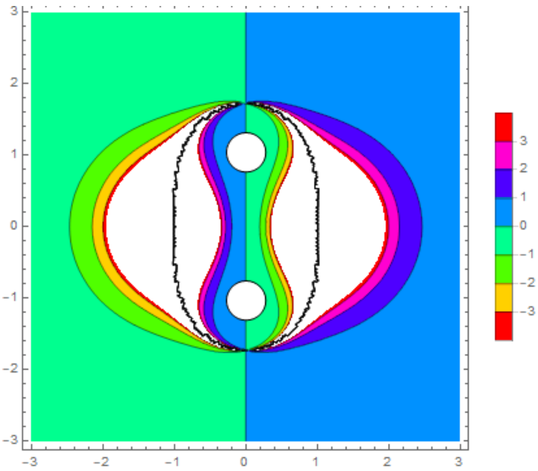

Figure 8. Plot of the non-dimensional pressure function in the *xy* plane for $\alpha = 2.0$.

Case (II): The case of two spheres of an unequal radius.

Following the process detailed above in case (I), we plot streamlines to understand the flow profile in unequal spheres. Figure 9 presents the streamline pattern for $\xi_1 = 1.0$ and $\xi_2 = -2.0$, and Fig. 10 presents that for $\xi_1 = 2.0$, $\xi_2 = -1.0$.

From Figs. 8 and 9, we observe that the flow is more disturbed when the smaller sphere faces the flow stream, i.e. when the smaller sphere is on the positive *x*-axis. It is also revealed in the pressure contours presented in Figs. 11 and 12. Further to this, we see from Fig. 12 that the region with higher pressure is larger (in area, blue shaded region) than the one in Fig. 11 (sea-blue shaded region).

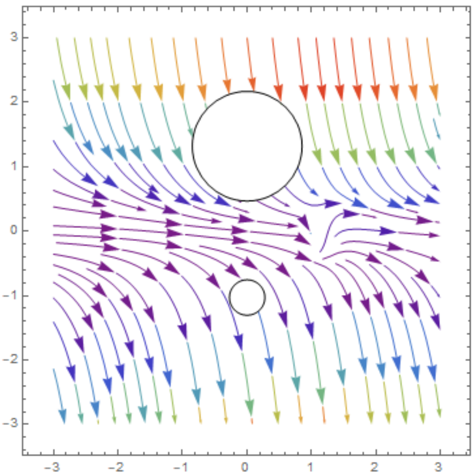

Figure 9. Streamlines in the xy plane for $\xi_1 = 1.0$, $\xi_2 = -2.0$.

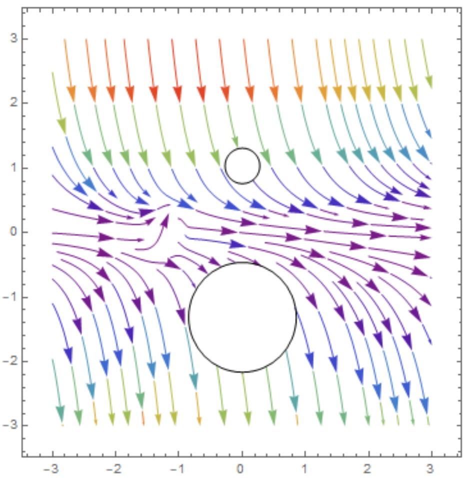

Figure 10. Streamlines in the xy plane for $\xi_1 = 2.0$, $\xi_2 = -1.0$.

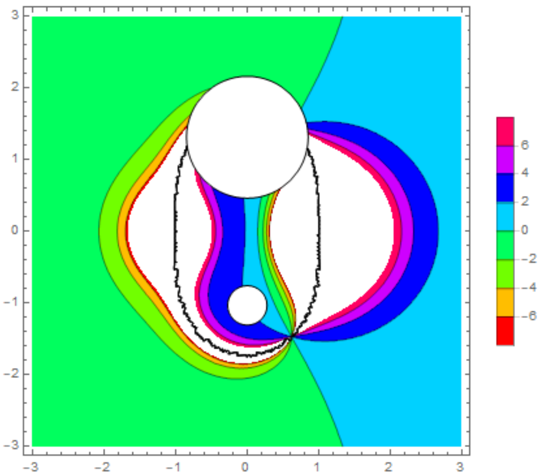

Figure 11. Plot of the non-dimensional pressure function in the xy plane for $\xi_1 = 2.0$, $\xi_2 = -1.0$.

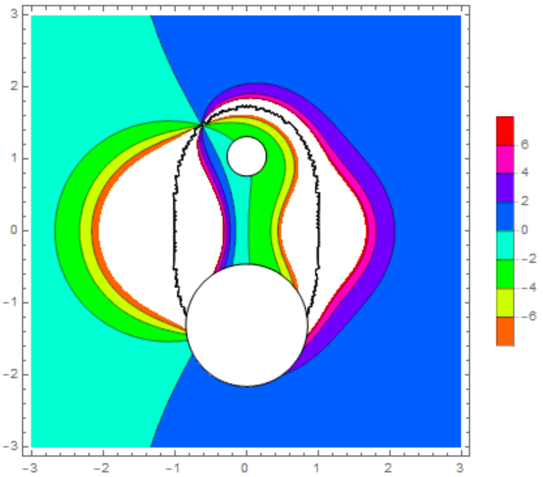

Figure 12. Plot of the non-dimensional pressure function in the xy plane for $\xi_1 = 1.0$, $\xi_2 = -2.0$.

CONCLUSION

The problem of Stokes' flow of viscous fluid past a pair of separated solid spheres (two-sphere) solved by Payne and Pell has been revisited in the present work. The formula for drag derived in terms of stream function is extended to compute the drag experienced by a system of two spheres. We derived the solution in the bipolar coordinate system and obtained expressions for drag and pressure. Further, we carried out a numerical evaluation of the stream function and depicted the flow pattern, drag, and pressure by varying the spheres' radii. Knowing the drag on spheres' system, we worked on computing bounds for the drag on individual spheres. Also, as a limiting case, we derived the drag experienced by a single sphere. These values match precisely with the ones found in literature.

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APPENDIX

Substituting the expression for pressure from Eq.(54) in Eq.(8), we have

$$\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sinh\xi\sum_{n=0}^{\infty}\left(G_n\cosh\left(n+\frac{1}{2}\right)\xi+H_n\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau)+(\cosh\xi-\tau)^{1/2}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left(G_n\sinh\left(n+\frac{1}{2}\right)\xi+H_n\cosh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau)=$$

$$=\frac{\mu}{a}\left(\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau)+(\cosh\xi-\tau)^{1/2}\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta'_{n+1}(\tau)\right) \quad \text{A.1}$$

Multiplying on both sides by $(\cosh\xi - r)^{-1}$ and integrating the resulting equation with respect to τ between the limits –1 and 1 gives,

$$\frac{1}{2}\sinh\xi\sum_{n=0}^{\infty}\left(G_n\cosh\left(n+\frac{1}{2}\right)\xi+H_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{P_n(\tau)}{(\cosh\xi-\tau)^{3/2}}d\tau+\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)\left(G_n\sinh\left(n+\frac{1}{2}\right)\xi+H_n\cosh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{P_n(\tau)}{(\cosh\xi-\tau)^{1/2}}d\tau=$$
$$=-\frac{\mu}{2a}\left(\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\vartheta_{n+1}(\tau)}{(\cosh\xi-\tau)^{3/2}}d\tau\right). \qquad \text{A.2}$$

Using the formulae given in Eqs.(40) and (41), integrals in the above expression can be evaluated to find the equation involving the constants G_n, H_n, A_n, B_n.

Substituting the expression for pressure from Eq.(54) into Eq.(9), we have

$$-\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sum_{n=0}^{\infty}\left(G_n\cosh\left(n+\frac{1}{2}\right)\xi+H_n\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n(\tau)+(\cosh\xi-\tau)^{1/2}\sum_{n=1}^{\infty}\left(G_n\cosh\left(n+\frac{1}{2}\right)\xi+H_n\sinh\left(n+\frac{1}{2}\right)\xi\right)P_n'(\tau)=$$
$$=-\frac{\mu}{a(1-\tau^2)}\left(-\frac{1}{2}(\cosh\xi-\tau)^{-1/2}\sinh\xi\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau)+(\cosh\xi-\tau)^{1/2}\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)\left(A_n\sinh\left(n+\frac{1}{2}\right)\xi+\right.\right.$$
$$\left.\left.+B_n\cosh\left(n+\frac{1}{2}\right)\xi\right)\vartheta_{n+1}(\tau)\right). \qquad \text{A.3}$$

Multiplying on both sides by $(\cosh\xi - \tau)^{-1}$ and integrating the resulting equation with respect to τ between the limits –1 and 1 gives,

$$-\sum_{n=0}^{\infty}\left(G_n\cosh\left(n+\frac{1}{2}\right)\xi+H_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\frac{d}{d\tau}\left((1-\tau^2)P_n(\tau)\right)}{(\cosh\xi-\tau)^{1/2}}d\tau+\sum_{n=1}^{\infty}\left(G_n\cosh\left(n+\frac{1}{2}\right)\xi+H_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{(1-\tau^2)P_n'(\tau)}{(\cosh\xi-\tau)^{1/2}}d\tau=$$
$$=-\frac{\mu}{a}\left(-\frac{1}{2}\sinh\xi\sum_{n=1}^{\infty}\left(A_n\cosh\left(n+\frac{1}{2}\right)\xi+B_n\sinh\left(n+\frac{1}{2}\right)\xi\right)\int_{-1}^{1}\frac{\vartheta_{n+1}(\tau)}{(\cosh\xi-\tau)^{3/2}}d\tau+\sum_{n=1}^{\infty}\left(n+\frac{1}{2}\right)\left(A_n\sinh\left(n+\frac{1}{2}\right)\xi+B_n\cosh\left(n+\frac{1}{2}\right)\xi\right)\times\right.$$
$$\left.\times\int_{-1}^{1}\frac{\vartheta_{n+1}(\tau)}{(\cosh\xi-\tau)^{1/2}}d\tau\right). \qquad \text{A.4}$$

Using the relations in Eqs.(40), (41), and the recurrence relations in Legendre polynomials, /4/, we can derive the expressions for G_n and H_n in terms of A_n and B_n. Again using the relations given in Eq.(35), these can in turn be written in terms of C_n, D_n, E_n, *and* F_n.