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APPLICATION OF THE FINITE ELEMENT METHOD TO VIBRATION ANALYSIS OF PIPES CONVEYING FLUID FLOW

PRIMENA METODE KONAČNIH ELEMENATA U ANALIZI VIBRACIJA CEVOVODA ZA TRANSPORT TEČNOSTI

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- natural frequency
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- FEM
- Matlab
- instability

Ključne reči
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- prirodne frekvencije
- elastičnost
- MKE
- Matlab
- nestabilnost

Abstract

In this paper, the equations are discretized with standard finite element method (FEM), and are generalized to analyse the free vibration problem of clamped-clamped pipes carrying fluid flow with several parameters. The flow circulating has the flexional motion as that of pipe structure. We developed a program under Matlab. The advantage of Matlab language by using standard functions is to present the first proper-modes of the system aspect interaction fluid-structure for different parameters in complex planes. The numerical approach is based on some research and analytical models. Numerical results show the effect of mass ratio, length, pressure force and Winkler elastic foundation on instabilities regions and static instability range.

Izvod

U radu se date jednačine diskretizuju standardnom metodom konačnih elemenata (MKE), a generalisane za analizu problema slobodnih vibracija obostrano uklještenih cevovoda za transport tečnosti sa nekoliko parametara. Tok strujanja ima oblik zavojnog kretanja, kao i sama konstrukcija cevovoda. Program je razvijen u Matlab. Prednost Matlab pri korišćenju standardnih funkcija je u zadavanju prvih pravilnih modova interakcije sistema sa aspekta fluid-konstrukcija za različite parametre u kompleksnim ravnima. Numerički pristup se zasniva na istraživačkim i analitičkim modelima. Numerički rezultati pokazuju uticaj odnosa mase, dužine, sile pritiska i Vinkler elastične podloge na nestabilne oblasti i opseg statičke nestabilnosti.

INTRODUCTION

The studies on the dynamic problem of pipe carrying fluid date back more than 60 years (Housner, 1952 /1/; Gregory and Païdoussis, 1966 /2/; Païdoussis 1966 /3/). Their studies were not extensive and in-depth; rather, limited to methods of solving this type of problem. Païdoussis' research continued for successive years until he was able to derive the motion equation for vibration of fluid-conveying pipe, which was done in 1977, /4/. In this, he followed an approximate analytical method called the Galerkin method. The equation was developed according to the physical and geometrical parameters affecting the stability of such type of system. These axes and studies were collected in a book of an old /5/, and recent publication /6/.

These articles and books are the starting point of many researchers. Chellapilla et al. /7/ studied the effect of a Pasternak foundation on the critical velocity of a fluid-conveying pipe by Galerkin method. He repeated the same search using fundamental frequencies calculations of a pipeline resting on a two-parameter foundation with different boundary conditions, /8/. Some studies have dealt with the thermal effect on instability such as that found by Qian et al. /9/, who studied the static instability of pinned-pinned fluid-conveying pipe under thermal loads. The equation of motion is derived for the straight pipe under the effects of linear and nonlinear stress-temperature cases. In addition to the analytical methods, the numerical methods presented significant results in studying this behaviour, similar to the method of spectral element modelling, /10/. Finite element analysis was used by Salah to analyse dynamically the stability of a pipe, stiffened by linear spring and conveying an internal flow of fluid, /11/. The same method was adopted in references /12-15/. This type of system was treated by numerical simulation, using the finite element method /16/. The results were similar to analytical results and a semi-analytical result, when calculating the first natural frequencies in terms of fluid velocity and for several parameters.

In the present research, calculation methods are developed for the analysis of stability regions, instability regions, and instability range in fixed-fixed pipe carrying fluid. Numerical modelling of pipe structure-fluid was conducted by finite element method /17, 18/. The characteristics of static and dynamic instability, namely the disappearance of the first vibratory mode and the attainment of the critical velocities associated with it, are carried out using a program developed in Matlab®. After studying the numerical approach, several examples are studied. We performed several calculations to obtain the critical velocities under various parameters, taking into account: fluid velocity, mass ratio, length, elastic foundation, and pressure force. The results are presented by displaying natural frequencies in terms of flow velocity, which enables us to analyse the instabilities under these effects.

DERIVATION OF GOVERNING DIFFERENTIAL EQUATION

The problem to be considered is the vibration analysis of a fluid conveying pipe system on an elastic foundation. The derivation of the equation is based on Euler-Bernoulli elementary beam theory. The physical model of conveying pipe carrying fluid is shown in Fig. 1a, Fig. 1b shows forces on fluid element, while Fig. 1c shows forces and moment of pipe element.

The pipe is long and straight L conveying an incompressible fluid with steady speed U; the motions are small δs.

The pipe rests on an elastic foundation Winkler-model soil of modulus KX, m_s and m_f are the masses per unit length of the pipe and the fluid, respectively. The boundary conditions for clamped-clamped pipe are,

$$Y\big|_{X=0} = \frac{\partial Y}{\partial X}\bigg|_{X=0} = Y\big|_{X=L} = \frac{\partial Y}{\partial X}\bigg|_{X=L} = 0\,. \qquad (1)$$

The equation for conveying pipe-carrying fluid on a Winkler elastic foundation is given as /5/,

$$EI\frac{\partial^4 Y}{\partial X^4} + (m_f U^2 + PA)\frac{\partial^2 Y}{\partial X^2} + 2m_f U\frac{\partial^2 Y}{\partial X \partial T} + $$
$$+(m_s + m_f)\frac{\partial^2 Y}{\partial T^2} + KY = 0\,. \qquad (2)$$

We use the same non-dimensional variables and parameters as in reference /6, 19/,

$$T = \left[\frac{EI}{(m_f + m_s)^{1/2}}\right]\frac{t}{L^2}\,,\quad \beta = \frac{m_f}{(m_f + m_s)}\,,$$

$$u = UL\left(\frac{m_f}{EI}\right)^{1/2}\,,\quad k = \frac{KL^2}{EI}\,,\quad \Pi = \frac{PAL^2}{EI}\,.$$

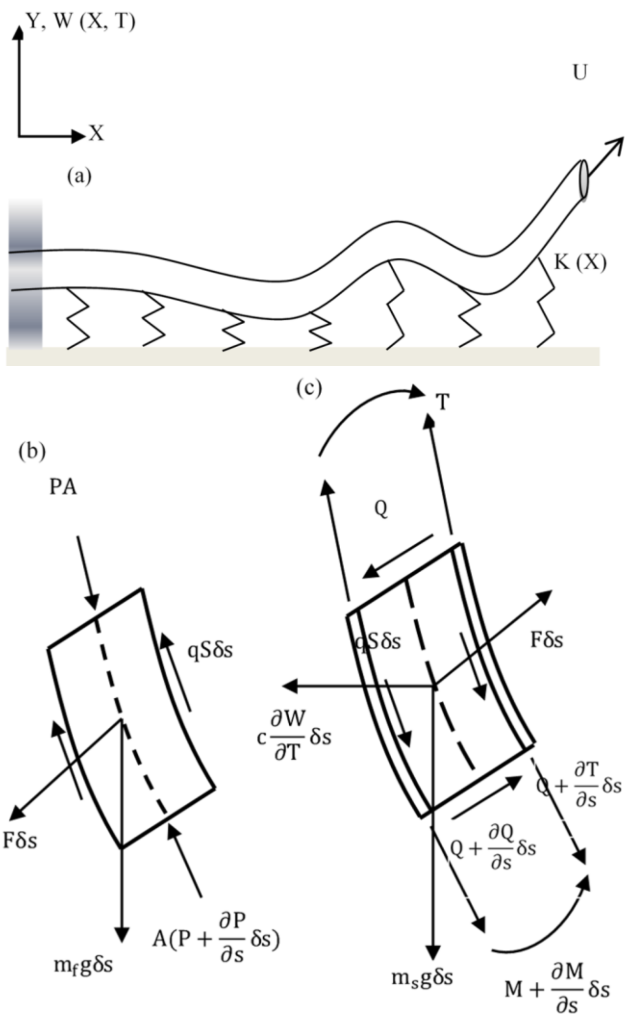


Figure 1. a) Representation of pipe-conveying fluid resting on an elastic Winkler-type; b) forces on fluid element; c) forces and moments on pipe element, δs, /6/.

FINITE ELEMENT DISCRETIZATION

The Eq.(2) is a fourth-order partial differential equation in two independent variables subject to various boundary conditions. It is not easy to get its analytical solution, but through the use of finite element method we get its numerical solution. The equation of element deflection for straight two-dimensional beam elements could have the form /20/,

$$W(X,T) = \sum_{(i=1)}^{N} N_i(X) W_i(T)\,, \qquad (3)$$

where: $[N_i]$ represent the shape function; $W_i(T)$ is the function which represents the shape of the displacements and rotations at nodes (Fig. 2).

Figure 2. Beam element nodal displacements.

Therefore, Eq.(3) becomes

$$W(X,T) = N_1(X)W_1(T) + N_2(X)\theta_1(T) + N_3(X)W_2(T) +$$
$$+N_4(X)\theta_2(T)\,, \qquad (4)$$

and

$$\theta(X,T) = N_1'(X)W_1(T) + N_2'(X)\theta_1(T) + N_3'(X)W_2(T) + N_4'(X)\theta_2(T)\,. \quad (5)$$

Elementary matrices

Potential energy of the solid element can be expressed,

$$V_s = \frac{1}{2}\int_0^L EI\left(\frac{d^2W}{dx^2}\right)^2 dx\,. \quad (6)$$

The kinetic energy of the solid element can be expressed,

$$T_s = \frac{1}{2}\int_0^L m_s \frac{d^2W}{dt^2} dx\,. \quad (7)$$

The kinetic energy of the fluid element /13/ can be expressed,

$$T_f = \frac{1}{2}\int m_f \left(U\frac{dW}{dx} + \frac{dW}{dt}\right)^2 dx\,. \quad (8)$$

The potential energy over the length of Winkler elastic foundation /15/ can be expressed,

$$V' = \frac{1}{2}\int_0^L KW^2 dx\,. \quad (9)$$

Different elementary matrices are given in appendix A.

After using the Lagrange principle, the equation of motion by finite element method is

$$[M]\{\ddot{q}\} + [C]\{\dot{q}\} + ([K])\{q\} = 0\,. \quad (10)$$

Analysis of dynamic eigenvalues

The governing equation of the system (structure plus fluid) can be transformed into its state-space coordinates,

$$E\dot{z} + Gz = 0\,, \quad (11)$$

where the state variable is

$$z = \begin{Bmatrix} \dot{q} \\ q \end{Bmatrix}. \quad (12)$$

The matrices [E] and [G] are calculated through variable changeset as the following,

$$E = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}, \quad (13)$$

$$G = \begin{bmatrix} C & K \\ -K & 0 \end{bmatrix}. \quad (14)$$

Therefore, we can obtain the natural frequencies (eigen-values) and mode shapes (eigen-vectors) by solving the mathematically well-known characteristic equation of

$$\lambda I - Hz = 0\,, \quad (15)$$

where: λ is eigen-value of the system; and I is a unity matrix, and

$$H = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}. \quad (16)$$

The solution of Eq.(10) can be written in the form,

$$\{q\} = \{E\}\exp(\lambda t)\,, \quad (17)$$

$$z = \begin{Bmatrix} \lambda\{E\} \\ \{E\} \end{Bmatrix}\exp(\lambda t) = \{\tilde{E}\}\exp(\lambda t)\,. \quad (18)$$

We obtain a homogeneous equation, which corresponds to a generalized eigen-value problem of our system,

$$\left\{\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} - \lambda\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right\}\begin{bmatrix} \lambda\{E\} \\ \{E\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (19)$$

We can compute the eigen-values numerically from Eq. (19) and obtain the eigen-frequencies of the conveying pipe carrying fluid for different various parameter values. The eigen-values are complex,

$$\lambda^m = Re^m + j\omega^m\,, \quad (20)$$

where: λ^m is the complex eigenvalue; Re is real part of eigen-frequencies; and the imaginary part of these roots represents the natural frequencies of damped system; and m = 1, 2 ….N.

RESULTS AND DISCUSSION

In the current research, results are discussed for various values of fluid velocity, mass ratio, length, elastic foundation (Winkler type), and pressure force, calculating the frequency of the first three eigen-modes for fluid-conveying pipe, with finding the critical velocities of instabilities.

The physical parameters are elastic modulus of structure is 211 GPa; incompressible fluid density is 1000 kg/m^3; elastic structure density is 7850 kg/m^3. The geometrical parameters are pipe length L belongs to [1, 2] m; the thickness corresponds to β [0.3, 0.7]; outer diameter of the pipe is 0.03 m. The research in our hands has adopted a beam with clamped-clamped boundary conditions. The cases can be divided into four; according to the parameters effect mass ratio β, length L, Winkler elastic foundation, and pressure effect.

Effect of mass ratio

In the case number one, firstly, the numerical results are obtained by differential transformation method (DTM) /22/, and FEM /17/ for mass ratio β = 0.1. The semi-analytical and numerical results are similar, see Fig. 3. We repeated the numerical calculations for different mass ratios, β = 0.3 in Fig. 4; β = 0.5 in Fig. 5, and β = 0.7, see Fig. 6.

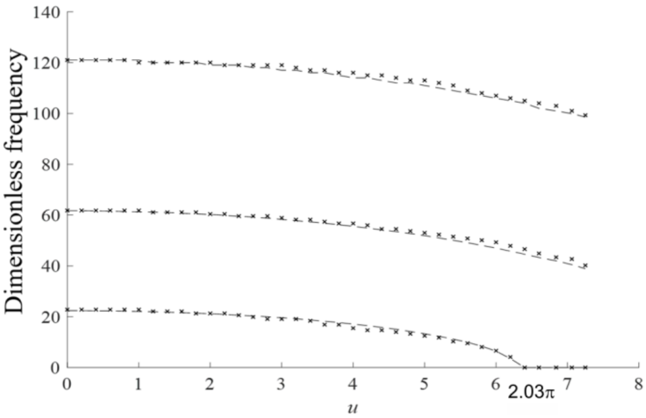


Figure 3. Dimensionless frequency for various values of *u*, for the lowest three modes of a clamped-clamped pipe, DTM /11/ (xxx), and FEM (− − −), β = 0.1.

In the same figures, part (a) presents the dimensionless results, and part (b) presents the dimensional results. The dimensional results show that there is a great variation in the natural frequency development, unlike what is found in the dimensionless frequencies, and this is contrary to many researchers, like Chellapilla et al. /7/. For U = 0 (no fluid flow), the first frequency variation is 3 %. For $U \equiv U_{cr}$, where the first critical velocity corresponds to static instability ($u = 2\pi$), we find the variation between the two cases equals 45 %, while the dynamic critical velocity is reduced by 43.60 %, corresponding to flutter. The variation of instability static range is 40 %. So, an increase in the value of β leads to a decrease in the stability region, critical velocities, as well as the instability range.

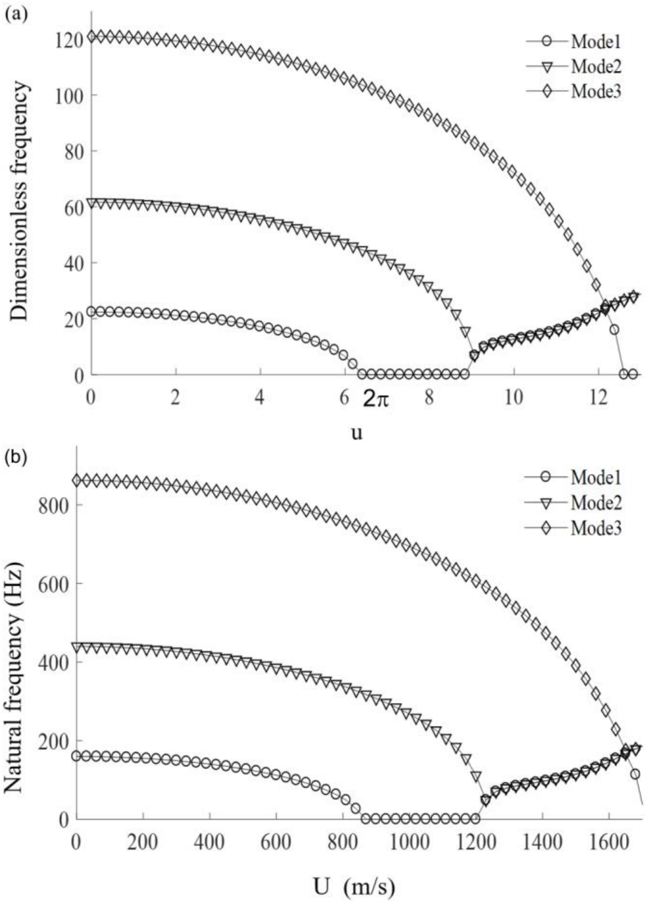
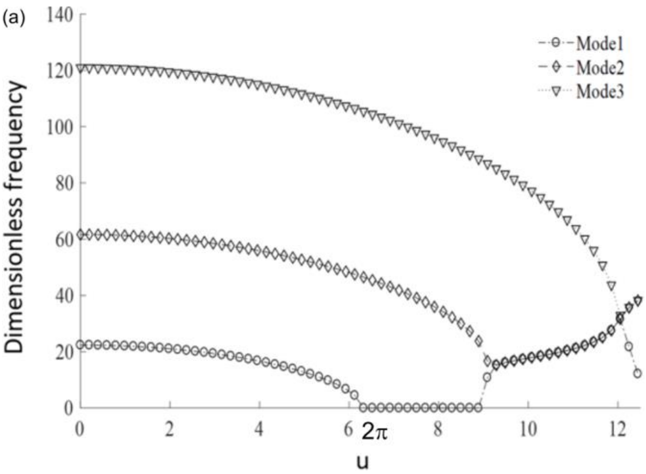


Figure 4. Three proper modes on fluid velocity function of clamped-clamped pipe carrying fluid, β = 0.3: a) dimensionless frequencies; b) naturel frequencies (Hz).



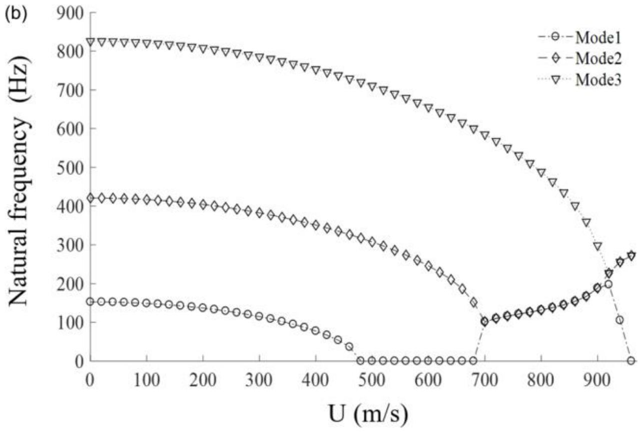


Figure 5. Three proper modes on fluid velocity function of clamped-clamped pipe carrying fluid, β = 0.5: a) dimensionless frequencies; b) naturel frequencies (Hz).

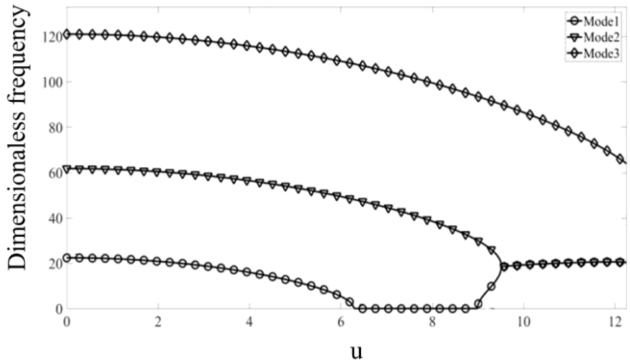


Figure 6. Three proper modes on fluid velocity function of clamped-clamped pipe carrying fluid, β = 0.7.

Effect of length and elastic foundation

In this case, we carry out the same study with the addition of the effect of elastic foundation and length. For elastic foundation Winkler-model we shall take two different values, a minimum value k = 10 and a maximum values k = 1000. Figures 7 and 8 present the first three natural frequencies as a function of the fluid velocity of clamped-clamped pipe on a weak elastic foundation Winkler-type, where k = 10, for three different lengths (L = 1, 1.5, and 2), with β = 0.3 and 0.5, respectively. For β = 0.3 and L = 1 (Fig. 7a), the largest change does not exceed 1 % for natural frequencies. While it exceeds 4 % for the instability range, the first critical velocity is 2.03π. Figure 7b presents a large variation in the natural frequencies, so it is 16%. The critical velocity corresponding to the buckling decreases to a limit of 5.41, i.e., 15.33 %, while the instability range value decreases to 36.8 %. When increasing the value of the length to double (L = 2), we find that the percentage of change in instability margin (range) diminishes to 40.9 % compared to the first case (L = 1). When we raise the value mass ratio to 0.5 as shown in Fig. 8. The instability range increases by 11 % with respect to the same length, see Fig. 8a, while the variation in the level of the first critical velocity and natural frequencies is almost non-existent. Figures 8b and 8c show the variations accompanying the increase in length. Figure 9a, b and c show the variation in the natural frequencies and critical velocities when increasing the value of elastic foundation to 1000 and with different lengths for β = 0.3.

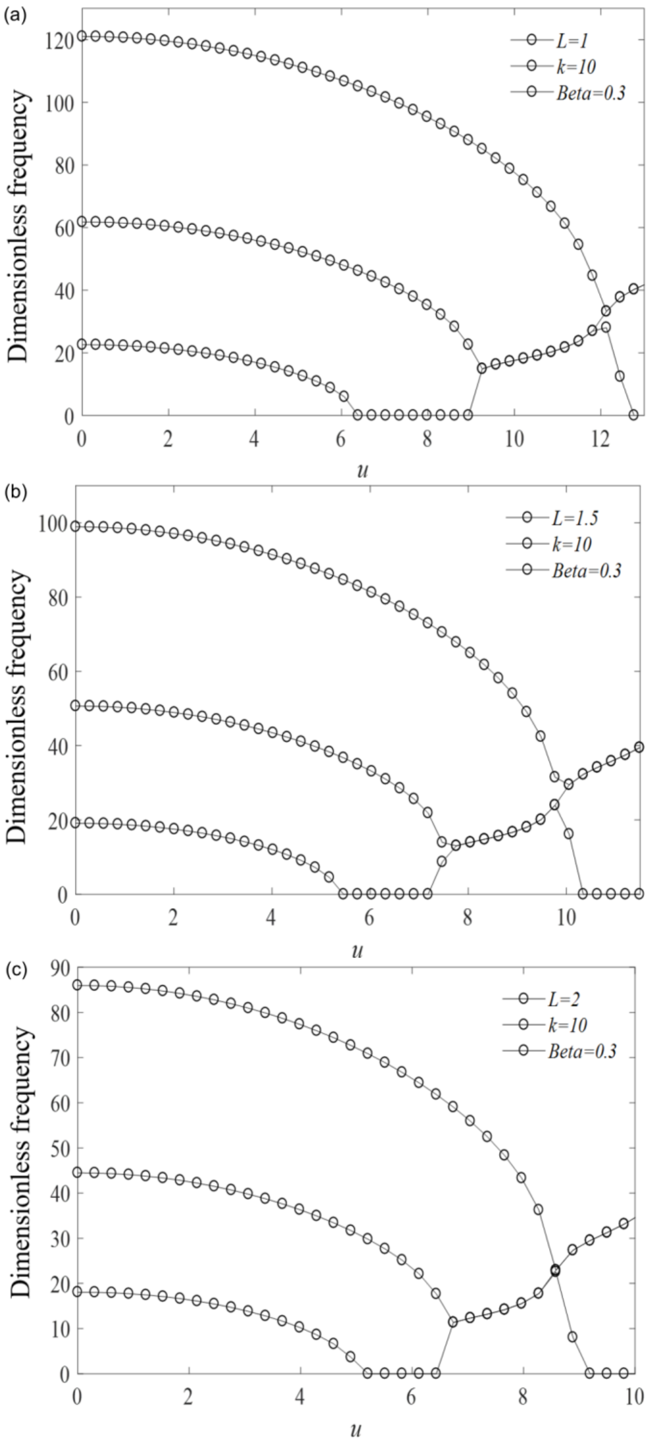


Figure 7. Effect of length on the natural frequency of the clamped-clamped pipe on elastic foundation (k = 10) at different fluid velocities, β = 0.3.

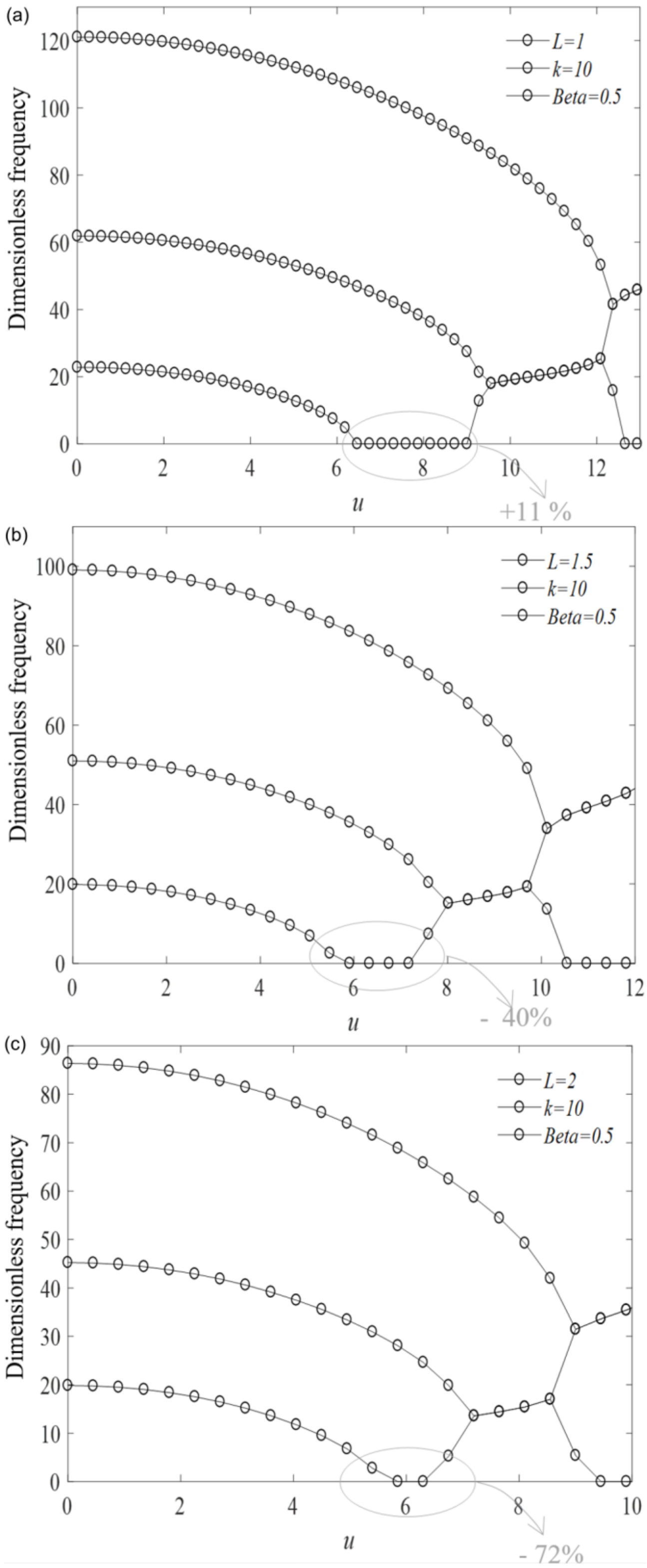


Figure 8. Effect of length on the natural frequency of the clamped-clamped pipe on elastic foundation (k = 10) at different fluid velocities, β = 0.5.

Figure 9a shows that the range of static instability deceased by 95 %, while the stability region expanded with the first critical velocity. We also note that the largest change in natural frequencies corresponds to the first natural frequency with an affinity ratio 41%. Otherwise, according to Fig. 9b, the frequencies decrease when increasing the length by about 150 %, where the highest percentage decrease is in the third frequency by 16.66 %.

Figures 9b and 9c show the disappearance of the instability range, which corresponds to the disappearance of dynamic instability.

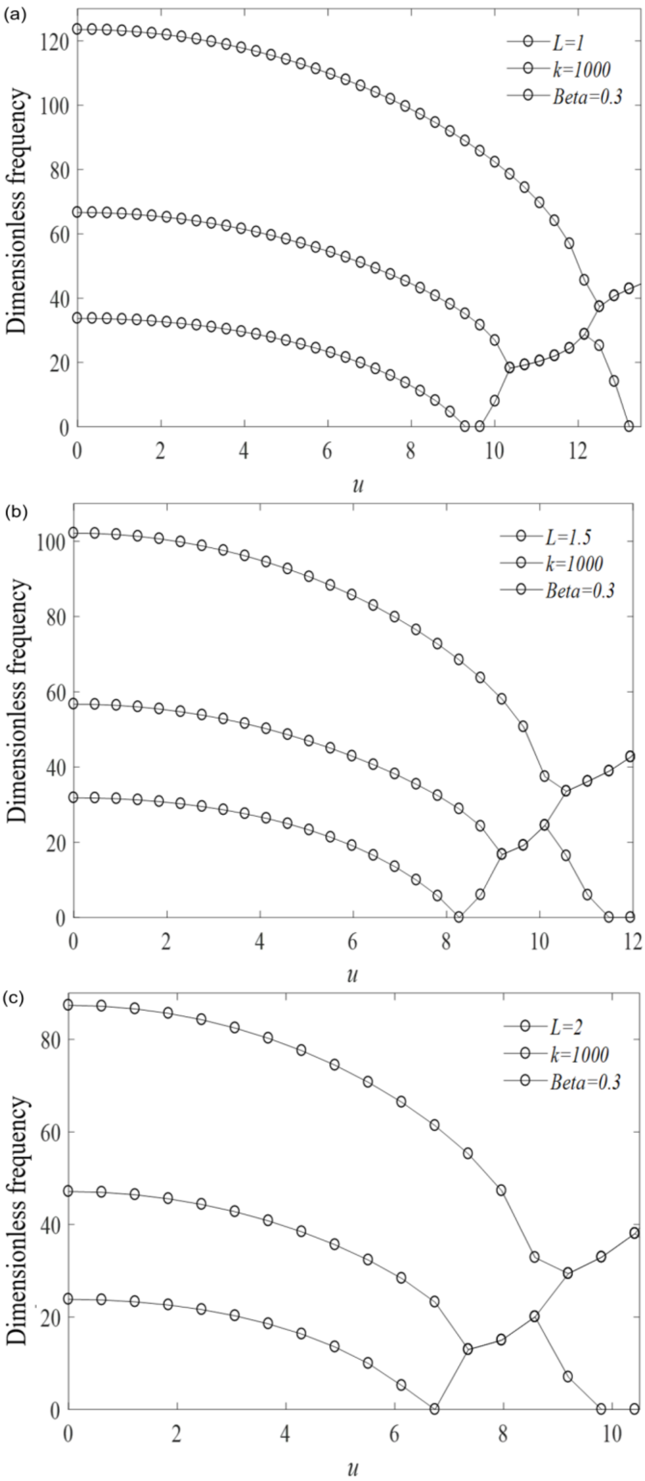


Figure 9. Effect of length on the natural frequency of the clamped-clamped pipe on elastic foundation (k = 1000) at different fluid velocities, β = 0.3.

In the same manner, the static critical velocity decreased, reducing stability region. Figure 10 provides roughly the same results as the Fig. 9c, as the mass ratio is 0.5.

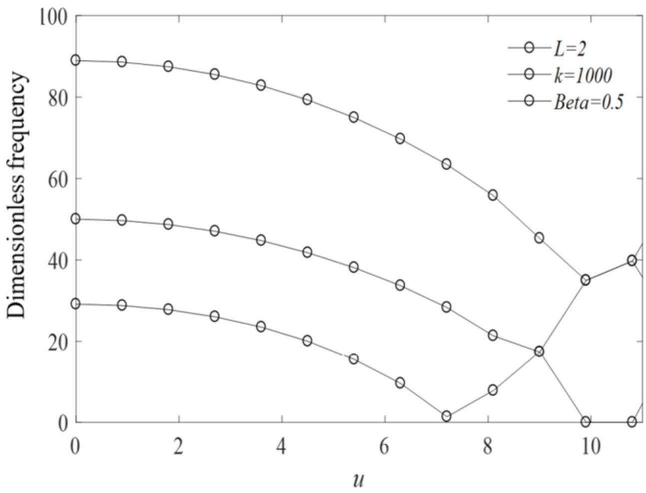


Figure 10. Effect of length on the natural frequency of the clamped-clamped pipe on elastic foundation (k = 1000) at different fluid velocities, β = 0.5.

Effect of pressure

At zero pressure (Π = 0), Figs. 3 to 6 give the following values of critical velocities as 2π for any value of β. As the pressure increases, the Figs. 11 to 14 show that the critical velocities decrease linearly. Excluding is the part of curve on domain [22 42] to Fig. 14, where the critical velocity takes a nonlinear curve. This is due to the effect of increasing the compressive force exerted on pipe ends as the pressure increases.

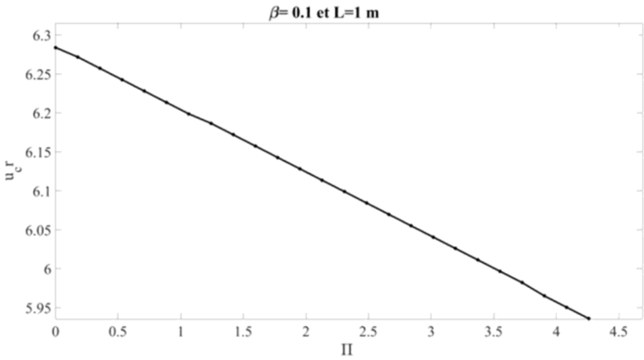


Figure 11. Critical velocities of buckling at different pressures of clamped-clamped pipe, β = 0.1.

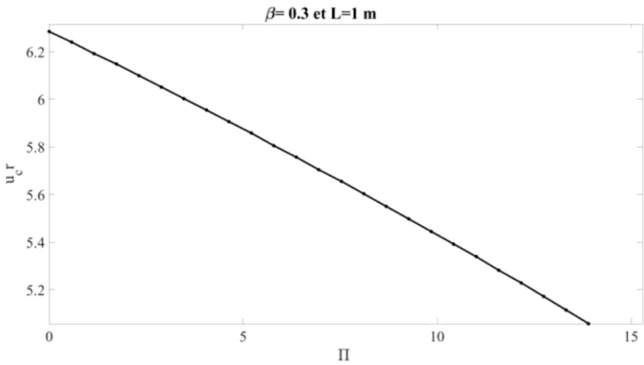


Figure 12. Critical velocities of buckling at different pressures of clamped-clamped pipe, β = 0.3.

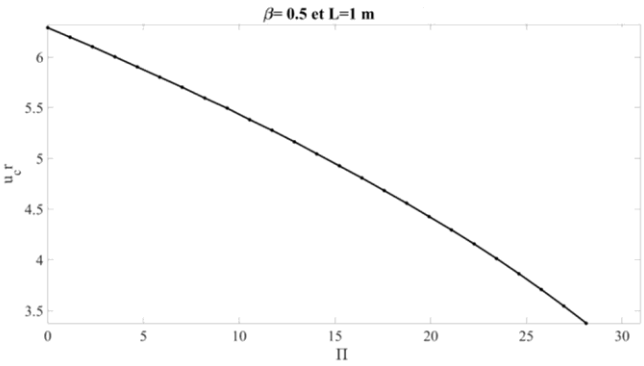


Figure 13. Critical velocities of buckling at different pressures of clamped-clamped pipe, β = 0.5.

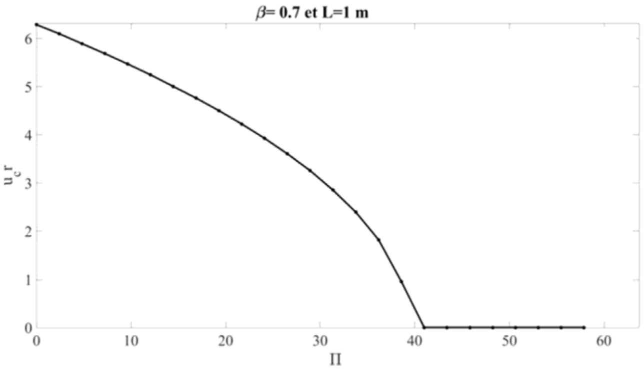


Figure 14. Critical velocities of buckling at different pressures of clamped-clamped pipe, β = 0.7.

These developments are mainly due to the effect of increasing the pressure force exerted on the ends of the pipes according to β. These are identical to the Euler load of buckling for the corresponding beams with compressive axial force. Figure 13 shows that the critical velocity is null when the pressure force reaches a value of 28, while its value approaches 42 with respect to β = 0.7, see Fig. 14.

CONCLUSIONS

In this work, we have studied the static and dynamic instability of clamped-clamped pipe under internal fluid. The numerical approach with the finite element method gives solutions in a complex plane by determining the proper-modes; the numerical results are finally combined with the semi-analytical results (DTM) to determine the different characteristics of instability of each system. The numerical study allowed us to obtain a very good precision by using the element of beam; each node contains two degrees of freedom. Several examples have been treated for the study of the influence of different geometrical and physical parameters on the system instability. From the discussions of numerical results of vibration, parametric instability for pipes conveying fluid, the main conclusions are presented:

1. we observe that instability appears when the velocity exceeds a threshold called critical velocity of instability, when the first natural pulse disappears;
2. the results obtained numerically are similar to those obtained by semi-analytical method for the determination of the first natural frequencies;
3. the natural frequencies decrease with increasing of fluid velocity;
4. the critical fluid flow velocity varies as a function of the mass ratio, and reflects the stability region of the system;
5. the Winkler type elastic foundation increases the rigidity of the system and therefore the critical instability velocities, while the range of static instability is decreasing;
6. we have noticed that increasing length L slightly decreases the natural frequencies of the system and consequently decreases their critical velocities, and leads to an increase in the instability range;
7. increasing the pressure decreases the critical velocities of buckling of conservative pipes.

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APPENDIX

The different elementary matrices for our system can be represented as follows, /17/,

$$[K_s]=\frac{m_f U^2}{30L}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, \quad (21)$$

$$[K_f]=\frac{m_f U^2}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & 3L^2 & -3L & 4L^2 \end{bmatrix}, \quad (22)$$

$$[M]=\frac{(m_s+m_f)L}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}, \quad (23)$$

$$[C]=\frac{2m_f U}{30}\begin{bmatrix} -30 & 6L & 30 & -6L \\ -6L & 0 & 6L & -L^2 \\ -30 & -6L & 30 & 6L \\ 6L & L^2 & -6L & 0 \end{bmatrix}, \quad (24)$$

$$[F]=\frac{KL}{420}\begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}, \quad (25)$$

$$[K_p]=\frac{PA}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & 3L^2 & -3L & 4L^2 \end{bmatrix}. \quad (26)$$