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ELASTIC-PLASTIC TRANSITION IN AN ORTHOTROPIC MATERIAL DISK
ELASTIČNO-PLASTIČNO PRELAZNO STANJE U DISKU OD ORTOTROPNOG MATERIJALA

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Ključne reči
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- brzina

Abstract

The purpose of this paper is to present a study of elastic-plastic transition in an orthotropic material disk by using transition theory. It has been observed that the isotropic material disk at the transition state requires more angular speed to yield at the internal surface in comparison to an orthotropic material disk. It has also been observed that an isotropic material disk at the transition state requires more angular speed to yield at the internal surface in comparison to an orthotropic material disk. The value of the hoop stress component is maximum at the r_i and magnitudes of hoop stress components are higher than those of radial stress components.

Izvod

Cilj ovog rada je predstavljanje studije elastoplastičnog prelaznog stanja kod diska od ortotropnog materijala primenom teorije prelaznih napona. Uočava se da disk od izotropnog materijala u prelaznom režimu zahteva veću ugaonu brzinu za pojavu tečenja na unutrašnjoj površini, u poređenju sa diskom od ortotropnog materijala. Uočava se da disk od izotropnog materijala u prelaznom režimu zahteva veću ugaonu brzinu za pojavu tečenja na unutrašnjoj površini, u poređenju sa diskom od ortotropnog materijala. Veličina komponente obimnog napona ima maksimum na r_i, a vrednosti komponente obimnog napona su veće od komponente napona u radijalnom pravcu.

INTRODUCTION

Theoretical investigation of elastic-plastic deformation in a disk, induced by centrifugal forces, is an important topic due to its various applications in engineering components such as gas turbines, flywheels, turbojet engines, reciprocating engines, centrifugal compressors, brake disks, etc. In this context, numerical investigations have been extensively used to predict the deformation, failure, and stress and strain fields in a uniform rotating disk under different loading conditions. Analytical solutions of the elastic-plastic problems in the rotating disks for isotropic materials can be found in many textbooks /1, 2, 5, 6/ and the stress analysis in the curvilinear orthotropic disk and cylinders under pressure can also be found in the literature, /8/. Arya et al. /7/ solved the problem of creep analysis of rotating orthotropic disks by using a method of successive approximations. Genta et al. /9/ have analysed stress distribution in orthotropic rotating disks. Jain et al. /10/ analysed rotating anisotropic disk of uniform strength, Sing et al. /11/ investigated the problem of creep deformation in orthotropic aluminium-silicon carbide composite rotating disk by using Hill's anisotropic yield. Hasan /12/ analysed thermal load in an orthotropic rotating disk. Saad et al. /13/ analysed the problem of elastic analysis in polar orthotropic FGM rotating disk for a variation function with parameters by using Hosford's yield criteria. Thakur et al. /18/ solved the problem of stress analysis in an orthotropic shell under thermal condition by using transition theory. The main objective of the present paper is to develop a consistent analytical model capable to resolve a class of control problems for the rotating disk, due to the importance of the control problems for rotating disks. On the other hand, the importance of material properties in the burst speed of disks is investigated. The novelty in the current research is to include control factors such as rotating speed in the consideration of the optimal performance of the disk.

MATHEMATICAL MODEL AND GOVERNING EQUATIONS

Consider an orthotropic material disk with constant thickness and density parameter, rotating with angular speed ω about an axis perpendicular to its plane and passing through the centre. Let the inner and outer radii of the disk in the deformed state be a and b, respectively, as shown in Fig. 1.

Displacement coordinates: the displacement components in cylindrical polar coordinates are given as:

$$u = r(1-\eta)\ ; \quad v = 0\ ; \quad w = dz\ , \tag{1}$$

where: η is position function, depending on the value of $r = \sqrt{(x^2 + y^2)}$ only, and d is a constant.

Generalized components of strain: the generalized components of strain are given as /3, 4/:

$$\varepsilon_{rr}=\frac{1}{n}\left[1-(r\eta'+\eta)^n\right],\quad \varepsilon_{\theta\theta}=\frac{1}{2}\left[1-\eta^n\right],$$

$$\varepsilon_{zz}=\frac{1}{2}\left[1-(1-d)^n\right],\quad \varepsilon_{zz}=\frac{1}{2}\left[1-(1-d)^n\right],$$

$$\varepsilon_{r\theta}=\varepsilon_{\theta z}=\varepsilon_{zr}=0\,. \tag{2}$$

where: $\eta' = d\eta/dr$.

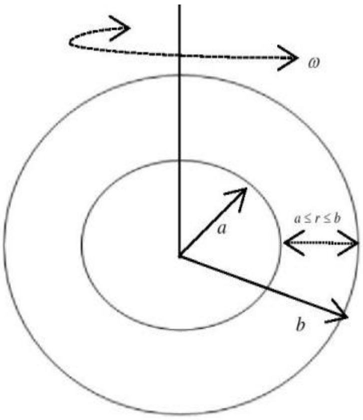

Figure 1. Geometry of orthotropic disk

Stress-strain relation: the constitutive equations for the orthotropic material are given as /2/:

$$\tau_{ij}=c_{ijkl}e_{kl}\,, \tag{3}$$

where: τ_{ij} and e_{kl} are stress and strain tensors; c_{ijkl} are the elastic constants, respectively. Substituting Eq.(2) into Eq.(3), we get:

$$\tau_{rr}=\frac{c_{11}}{n}\left[1-(r\eta'+\eta)^n\right]+\frac{c_{12}}{n}\left[1-\eta^n\right]+\frac{c_{13}}{n}\left[1-(1-d)^n\right],$$

$$\tau_{\theta\theta}=\frac{c_{21}}{n}\left[1-(r\eta'+\eta)^n\right]+\frac{c_{22}}{n}\left[1-\eta^n\right]+\frac{c_{23}}{n}\left[1-(1-d)^n\right],$$

$$\tau_{r\theta}=\tau_{\theta z}=\tau_{zr}=\tau_{zz}=0\,. \tag{4}$$

Equation of equilibrium: the equation of equilibrium is all satisfied except:

$$\frac{d}{dr}(r\tau_{rr})=\tau_{\theta\theta}-\rho\omega^2r^2\,, \tag{5}$$

where: τ_{rr} is radial stress; and $\tau_{\theta\theta}$ are hoop stresses; ρ is the density of the material.

Asymptotic analysis: using Eq.(4) in Eq.(5), we get a nonlinear differential equation with respect to η as:

$$\frac{d\eta}{dT}\left[T(1+T)^n+\frac{c_{12}}{c_{11}}T-\frac{1}{n\eta^n c_{11}}\{(c_{11}-c_{21})[1-\eta^n(T+1)^n]+(c_{12}-c_{22})\times\right.$$

$$\left.\times(1-\eta^n)+(c_{13}-c_{23})[1-(1-d)^n]\}-\frac{\rho\omega^2r^2}{c_{11}\eta^n}\right]+\eta T(1+T)^{n-1}=0 \tag{6}$$

where: $r\eta' = \eta T$.

Transition points: the transition points of η in Eq.(6) are $T\to 0$, -1 and $T\to\pm\infty$. $T\to 0$ gives nothing of importance.

Boundary condition: boundary conditions of the problem are given by:

$$\tau_{rr}=0\quad \text{at}\quad r=a\quad \text{and}\quad r=b\,, \tag{7}$$

where: τ_{rr} denotes the stress along radial direction, respectively.

PROBLEM SOLUTION

For finding the plastic stress distribution, the asymptotic analysis through the principal stress leads from elastic state to plastic state /3, 4, 14-19/ at the turning point $T\to\pm\infty$. Then we define the transition function Z as:

$$Z=1-\frac{n\tau_{rr}}{c_{11}+c_{12}+c_{13}}-\frac{n\rho\omega^2r^2}{2(c_{11}+c_{12}+c_{13})}\,, \tag{8}$$

where: Z is a function of r only. By taking logarithmic differentiation from Eq.(8) with respect to r and using Eq.(4) and Eq. (6), we get:

$$\frac{d\ln Z}{dT}=\frac{1}{r(c_{11}+c_{12}+c_{13})}\times \tag{9}$$

$$\times\frac{(c_{11}-c_{21})[1-\eta^n(T+1)^n]+(c_{12}-c_{22})(1-\eta^n)+(c_{13}-c_{23})[1-(1-d)^n]}{\eta^n(T+1)^n+\frac{c_{12}}{c_{11}}\eta^n+\frac{c_{13}}{c_{11}}(1-d)^n-\frac{n\rho\omega^2r^2}{2c_{11}}}$$

By taking the asymptotic value of Eq.(9) as $T\to\pm\infty$ and integrating, we get

$$Z=A_0r^{-\frac{(c_{11}-c_{21})}{c_{11}}}\,, \tag{10}$$

where: A_0 is a constant of integration. From Eqs.(8) and (10), we get

$$\tau_{rr}=\frac{(c_{11}+c_{12}+c_{13})}{n}\left[1-A_0r^{-\frac{(c_{11}-c_{21})}{c_{11}}}\right]-\frac{\rho\omega^2r^2}{2}\,. \tag{11}$$

The relation between yielding stress in tension and elastic material constants at the transition range is given by /4/:

$$Y=\frac{c_{11}c_{22}c_{33}-c_{11}c_{23}^2-c_{33}c_{12}^2+2c_{12}^2c_{23}-c_{13}^2c_{22}}{n(c_{22}c_{33}-c_{23}^2)}\,,$$

where: Y is the yielding stress. Substituting the values Y in Eq.(11), we get

$$\tau_{rr}=KY\left[1-A_0r^{-T}\right]-\frac{\rho\omega^2r^2}{2}\,, \tag{12}$$

where: $K=\frac{(c_{22}c_{33}-c_{23}^2)(c_{11}+c_{22}+c_{33})}{c_{11}c_{22}c_{33}-c_{11}c_{23}^2-c_{33}c_{12}^2+2c_{12}^2c_{23}-c_{13}^2c_{22}}$ and $T=\frac{c_{11}-c_{21}}{c_{11}}$.

Substituting Eq.(12) into Eq.(5), we get:

$$\tau_{\theta\theta}=KY\left[1-A_0r^{-T}(1-T)\right]-\frac{\rho\omega^2r^2}{2}\,. \tag{13}$$

Substituting Eq.(7) into Eq.(12), we obtain

$$A_0=\left[1-\frac{\rho\omega^2a^2}{2KY}\right]a^T\quad \text{and}\quad \frac{\rho\omega^2}{2}=\frac{(a^{-T}-b^{-T})KY}{(a^{-T}b^2-a^2b^{-T})}\,, \tag{14}$$

where: $K=\frac{(c_{22}c_{33}-c_{23}^2)(c_{11}+c_{22}+c_{33})}{c_{11}c_{22}c_{33}-c_{11}c_{23}^2-c_{33}c_{12}^2+2c_{12}^2c_{23}-c_{13}^2c_{22}}$.

Substituting Eq.(14) into Eqs.(12) and (13), we get

$$\tau_{rr}=\frac{\rho\omega^2}{2}\left[\frac{a^{-T}(b^2-r^2)+b^{-T}(r^2-a^2)+r^{-T}(a^2-b^2)}{a^{-T}-b^{-T}}\right], \quad (15)$$

$$\tau_{\theta\theta}=\frac{\rho\omega^2}{2}\left[\frac{a^{-T}(b^2-r^2)+b^{-T}(r^2-a^2)+r^{-T}(1-T)(a^2-b^2)}{a^{-T}-b^{-T}}\right].$$

Initial yielding: it has been seen that from Eq.(15) the $|\tau_{\theta\theta}|$ is maximum at $r=\left[\frac{T(T-1)(a^2-b^2)}{2(a^{-T}+b^{-T})}\right]^{1/2+T}=r_i$ (say intermediate zone), depending upon the values of elastic constants c_{11}, c_{12}, c_{13} and c_{21}. Therefore, yielding of the disk will take place at $r = r_i$ and Eq.(15), the hoop stress can be written as:

$$\left|\tau_{\theta\theta}\right|_{r=r_i}=\left|\frac{\rho\omega^2}{2}\left[\frac{a^{-T}(b^2-r_i^2)+b^{-T}(r_i^2-a^2)+r_i^{-T}(1-T)(a^2-b^2)}{a^{-T}-b^{-T}}\right]\right|=$$

$$=Y^* \text{ (say)}, \quad (16)$$

where: Y^* is the yielding stress.

Investigation of the angular speed for orthotropic disk: from Eq.(16), the angular speed required for initial yielding becomes:

$$\Omega^2_{\text{inital yielding}}=\frac{\rho\omega^2b^2}{Y^*}= \quad (17)$$

$$=\text{mod}\frac{2\left[(a/b)^{-T}-1\right]}{(a/b)^{-T}\left(1-K^{\frac{2}{2+T}}\right)-\left[K^{\frac{2}{2+T}}-(a/b)^2\right]+K^{-\frac{T}{2+T}}(1-T)\left[(a/b)^2-1\right]}$$

where: $K=\frac{T(T-1)\left[(a/b)^2-1\right]}{(a/b)^{-T}+1}$; and $\omega_i=\frac{\Omega_i}{b}\sqrt{\frac{Y^*}{\rho}}$.

Non-dimensional components: we introduce the following non-dimensional components as: $R = r/b$, $R_0 = a/b$, $\sigma_r = \tau_{rr}/Y^*$, $\sigma_\theta = \tau_{\theta\theta}/Y^*$; and $\Omega_i^2 = \rho\omega_i^2b^2/Y^*$. Elastic-plastic transitional stresses and angular speed from Eqs.(15) and (17) become:

$$\sigma_r=\frac{\Omega^2_{\text{inital yielding}}}{2}\left[\frac{R_0^{-T}(1-R^2)+(R^2-R_0^2)+R^{-T}(R_0^2-1)}{R_0^{-T}-1}\right],$$

$$\sigma_\theta=\frac{\Omega^2_{\text{inital yielding}}}{2}\left[\frac{R_0^{-T}(1-R^2)+(R^2-R_0^2)+R^{-T}(1-T)(R_0^2-1)}{R_0^{-T}-1}\right],$$

$$\Omega^2_{\text{inital yielding}}= \quad (18)$$

$$=\frac{2(R_0^{-T}-1)}{R_0^{-T}\left(1-K_1^{\frac{2}{2+T}}\right)-\left(K_1^{\frac{2}{2+T}}-R_0^2\right)+K_1^{-\frac{T}{2+T}}(1-T)(R_0^2-1)},$$

where: $K_1=\frac{T(T-1)(R_0^2-1)}{2(R_0^{-T}+1)}$.

Fully-plastic state: Seth /4/ investigated the following relationship between constants for fully-plastic state, given as: $c_{11} = c_{13} = c_{12}$; $c_{21} = c_{23} = c_{22}$; $c_{31} = c_{32} = c_{33}$. Equation (18) for fully plastic state becomes:

$$\sigma_r=\frac{\Omega^2_{\text{fully plastic}}}{2}\left[\frac{R_0^{-T_1}(1-R^2)+(R^2-R_0^2)+R^{-T_1}(R_0^2-1)}{R_0^{-T_1}-1}\right], \quad (19)$$

$$\sigma_\theta=\frac{\Omega^2_{\text{fully plastic}}}{2}\left[\frac{R_0^{-T_1}(1-R^2)+(R^2-R_0^2)+R^{-T_1}(1-T_1)(R_0^2-1)}{R_0^{-T_1}-1}\right],$$

where: $K_2=\frac{T_1(T_1-1)(R_0^2-1)}{2(R_0^{-T_1}+1)}$; and $T_1=\frac{c_{11}-c_{22}}{c_{11}}$.

Isotropic case: the material constants reduce to two constants only, for an isotropic case, /2/, and Eq.(18) becomes:

$$\sigma_r=\frac{\Omega^2_{\text{isotropic inital yielding}}}{2}\left[\frac{R_0^{-c}(1-R^2)+(R^2-R_0^2)+R^{-c}(R_0^2-1)}{(R_0^{-c}-1)}\right]$$

$$\sigma_\theta=\frac{\Omega^2_{\text{isotropic inital yielding}}}{2}\left[\frac{R_0^{-c}(1-R^2)+(R^2-R_0^2)+R^{-c}(1-c)(R_0^2-1)}{(R_0^{-c}-1)}\right]$$

$$\Omega^2_{\text{isotropic inital yielding}}= \quad (20)$$

$$=\frac{2(R_0^{-c}-1)}{R_0^{-c}\left(1-K_3^{\frac{2}{2+c}}\right)-\left(K_3^{\frac{2}{2+c}}-R_0^2\right)+K_3^{-\frac{c}{2+c}}(1-c)(R_0^2-1)},$$

where: $K_3=\frac{c(c-1)(R_0^2-1)}{2(R_0^{-c}+1)}$; and $c = c_{11} - c_{12}/c_{11} = T$.

Fully-plastic state for isotropic materials: the stresses and angular speed for fully plastic state ($c \to 0$) in Eq.(20) become:

$$\sigma_r=\frac{\Omega^2_{\text{isotropic fully-plastic}}}{2}\left[\frac{\ln R_0(1-R^2)+\ln R(R_0^2-1)}{\ln R_0}\right],$$

$$\sigma_\theta=\frac{\Omega^2_{\text{isotropic fully-plastic}}}{2}\left[\frac{\ln R_0(1-R^2)+(R_0^2-1)\ln R+R_0^2-1}{\ln R_0}\right],$$

and $$\Omega^2_{\text{isotropic fully-plastic}}=\left|\frac{2\ln R_0}{\ln R_0+R_0^2-1}\right|. \quad (21)$$

NUMERICAL RESULTS AND DISCUSSION

To illustrate the analysis, we have to take the numerical values of elastic constants from the existing literature. As a numerical example, the elastic constants are given for the orthotropic material such as barite ($BaSO_4$) /1/ as: c_{11} = 907, c_{12} = 273, c_{13} = 275, c_{23} = 468, c_{22} = 800, c_{33} = 1074, c_{44} = 122, c_{55} = 293, c_{66} = 283, respectively. Curves are drawn between angular velocity and various radii ratio R_0 = a/b (see Fig. 2) for a disk made of isotropic (i.e. c = 0.25, saturated clay; c = 0.5, copper) and orthotropic (i.e. $BaSO_4$, barite) materials at the transition and fully-plastic state.

It is observed from Fig. 2, that the isotropic material disk at the transition state requires more angular speed to yield at the internal surface in comparison to orthotropic material disk. For the full-plastic state (i.e. $c_{11} = c_{12} = c_{13} = c_{21} = c_{22} = c_{23} = c_{31} = c_{32} = c_{33} = 0$). It can be seen that isotropic material disk requires lesser angular speed to become fully plastic than the orthotropic material disk.

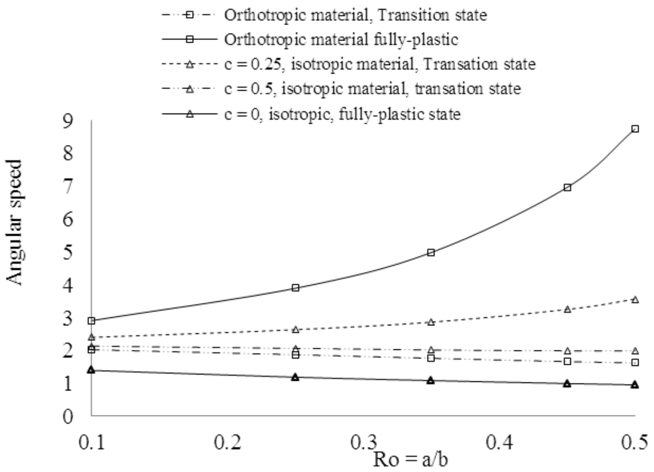

Figure 2. Transition yielding and fully-plastic state of the orthotropic and isotropic disks.

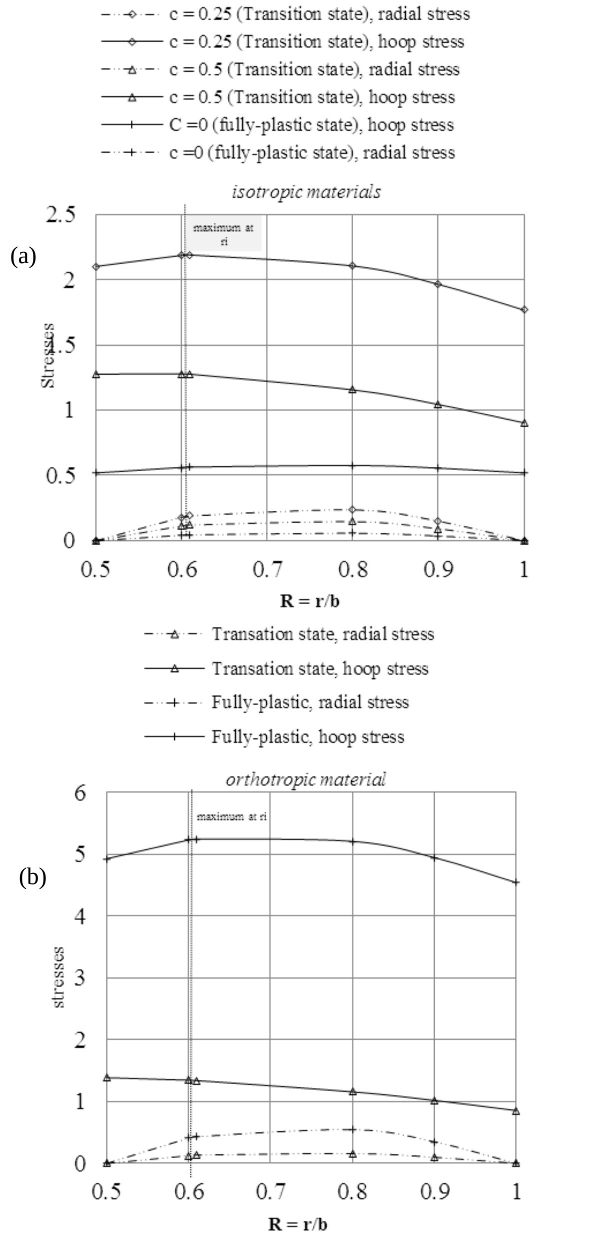

Figure 3. Stress distribution in transition and fully plastic states: a) isotropic, and b) orthotropic material, along radius ratio $R = r/b$.

From Fig. 3, curves are produced stresses vs. radii ratio ($R = r/b$). It is concluded that the hoop stress occurs at the inner surface for the disk of isotropic and orthotropic materials at the transition and fully-plastic state. The value of hoop stress component is maximum at the intermediate zone (i.e. at $r = r_i$), and the magnitudes of the radial stress component are lesser than the hoop stress component.

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