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LEBESGUE MEASURE IN AN ELASTO-PLASTIC SHELL

LEBESGOVA DEFORMACIJA U ELASTOPLASTIČNOJ LJUSCI

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- Lebesgova deformacija

Abstract

The purpose of this paper is to present the study of Lebesgue strain measure in an elasto-plastic shell. Considered is the problem of a spherical shell under internal and external pressure. Results are similar to the Tresca yield condition and Hulsurkar when the values of the Lebesgue measure are L(1) = 0.5 and L(1) = 1.

Izvod

Cilj ovog rada je predstavljanje rezultata merenja Lebesgove deformacije u elastoplastičnoj ljusci. Problem se bavi sfernom ljuskom opterećenom spoljašnjim i unutrašnjim pritiskom. Rezultati su slični kao i u slučaju uslova tečenja po Treski i Hulsurkaru, kada su vrednosti Lebesgove deformacije L(1) = 0.5 i L(1) = 1.

INTRODUCTION

Shells are important for various engineering applications. Analysis and design of these structures are therefore of continuing interest to the scientific and engineering communities. Accurate and conservative assessments of the maximum load carried by the structure, as well as the equilibrium path in both elastic and inelastic range are, therefore, of paramount importance in understanding the integrity of the structure. Spherical shells subjected to external hydrostatic pressure have been employed as structural components of undersea systems, flight vehicles, vacuum chambers; chambers fixed to the ocean floor and many other applications. For example, complete transparent spherical shells have been enclosed to house undersea transducers and used as video systems and have been proposed for carrying humans to significant depths. Incomplete spherical shells have found application as end caps of the cylinder-cone-sphere systems. Some of these applications have involved transparent partial spheres. Karman /1/ analysed the buckling of spherical shells by external pressure. Uddin /2/ investigated buckling of general spherical shells under external pressure. Gupta et al. /3/ discussed the problem of elastic-plastic transition in an orthotropic shell under internal pressure by using Seth transition theory. Blachut /4/ analysed composite spheroidal shells under external pressure. Thakur et al. /5/ investigated elastic-plastic stress analysis in a spherical shell under internal pressure and steady state temperature. Liu /6/ investigated the dynamic buckling of spherical shell structures due to subsea collisions. In this paper, we investigate the Lebesgue measure in an elasto-plastic shell by using Seth's transition theory. Seth's transition theory /7/ utilizes the concept of generalized strain measure and asymptotic solution at the critical points, or turning points of differential equations, defining the deformed field, and has been applied to a large number of problems successively (Gupta et al. /3, 8/, Seth /7, 9/, Thakur et al. /10-30, 32-35/.

Lebesgue strain measure: the Lebesgue strain measure in terms of the principal Almansi strain components is /8/:

$$\varepsilon_{ii}=\frac{1}{2}\int_0^{e}L'(1\text{-}2e_{ii}^{A})de_{ii}^{A}=\frac{1}{2}\Big[L(1)\text{-}L(1\text{-}2e_{ii}^{A})\Big], \qquad (1)$$

such that $L(0) = 0$, $L(\infty)$, $L'(0) = 0$.

GOVERNING EQUATIONS

Let us consider a spherical shell made of compressible and incompressible material, with the central bore of radius a and external radius b respectively. The thickness of the spherical shell is assumed to be constant.

Displacement coordinates: the components of displacement in spherical coordinates (r, θ, ϕ) are taken as:

$$u=r(1-\eta);\ \ v=0;\ \ w=0\,, \qquad (2)$$

where: u, v, w (displacement components); and η is a function of r.

Generalized strain components: the generalized components of strain are obtained from Eq.(1) as:

$$\begin{aligned} e_{rr}&=\frac{1}{n}\Big[L(1)-L(r\eta'+\eta)^2\Big],\\ e_{\theta\theta}&=\frac{1}{n}\Big[L(1)-L(\eta^2)\Big]=e_{\phi\phi},\\ e_{r\theta}&=e_{\theta\phi}=e_{\phi r}=0, \end{aligned} \qquad (3)$$

where: r, θ, ϕ be spherical coordinates; and η be the measure; and $\eta' = d\eta/dr$.

Stress-strain relation: the stress-strain relation for isotropic material is given /31/ as

$$\tau_{ij}=\lambda\delta_{ij}I_1+2\mu e_{ij};\ \ (i,j=1,2,3)\,, \qquad (4)$$

where: τ_{ij} and e_{ij} are stress and strain tensor; λ and μ are Lame's constants; δ_{ij} is Kronecker delta; and $I_1 = e_{kk}$ is the first strain invariant. Substituting Eq.(3) into Eq.(4), we get

$$\tau_{rr} = L(1)\left[\frac{3\lambda}{2}+\mu\right] - L\left\{(r\eta'+\eta)^2\right\}\left[\frac{\lambda}{2}+\mu\right] - \frac{\lambda}{2}L(\eta^2),$$

$$\tau_{\theta\theta} = \tau_{\phi\phi} = L(1)\left[\frac{3\lambda}{2}+\mu\right] - L(\eta^2)\left[\frac{\lambda}{2}+\mu\right] - \frac{\lambda}{2}L\left\{(r\eta'+\eta)^2\right\}; \quad (5)$$

$$\tau_{r\theta} = \tau_{\theta\phi} = \tau_{\phi r} = 0.$$

Equation of equilibrium: equations of equilibrium are /31/:

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{2(\tau_{rr}-\tau_{\theta\theta})}{r} = 0. \quad (6)$$

Critical points: by substituting Eq.(5) into Eq.(6), we obtain a nonlinear differential equation with respect to η as:

$$P\eta^2\frac{dP}{d\eta} = \frac{-2(1-C)\eta^2 PL'(\eta^2) + C\left[L(\eta^2) - L\left\{(r\eta'+\eta^2)\right\}\right]}{\eta(P+1)L'\left\{(r\eta'+\eta^2)\right\}nc_{11}\eta^n} - \quad (7)$$
$$-\eta P(P+1),$$

where: $r\eta' = \eta P$; and C is the compressibility factor in terms of λ and μ is given $C = 2\mu/(\lambda + 2\mu)$.

Transition points: the transition points of η in Eq.(7) are: $P = 0$, $P \to -1$, and $P \to \pm\infty$.

SOLUTION OF THE PROBLEM

For finding the plastic stress distribution, the transition function is taken through the principal stresses (see /4, 5, 7, 8-29, 32-35/) at the transition point $P \to \pm\infty$, we define the transition function $\Im$ as:

$$\Im = 1 - \frac{\tau_{rr}}{L(1)(2\mu+3\lambda)} = \frac{1}{L(1)(3-2C)}\left[L(r\eta'+\eta^2) - 2(1-C)L(\eta^2)\right] \quad (8)$$

where: $\Im$ be the transition function function of r only. Taking the logarithmic differentiation of Eq.(8) with respect to r and using Eq.(7), and taking the asymptotic value $P \to \pm\infty$, then after integration we get

$$\Im = A_1 r^{-2C}, \quad (9)$$

where: A_1 is a constant of integration. From Eq.(8) and Eq.(9), we have

$$\tau_{rr} = \frac{Y(2-C)}{C}L(1)\left[1 - A_1 r^C\right], \quad (10)$$

where: Y is the yielding stress and $\mu(3 - 2C) = Y(2 - C)$.

Case 1: Pressure applied at the internal surface

Let us consider a uniform pressure applied at the internal surface of the spherical shell, say p_i, and the boundary conditions are at $\tau_{rr} = -p_i$ at $r = a$ and $\tau_{rr} = 0$ and $r = b$ as shown in Fig. 1. Using boundary condition in Eq.(10), we get $A_1 = b^{2C}$. Now substituting the value of constant A_1 in Eq.(10) and using Eq.(6), we get

$$\tau_{rr} = Y\frac{(2-C)}{C}L(1)\left\{1 - \left(\frac{b}{r}\right)^{2C}\right\},$$

$$\tau_{\theta\theta} = \tau_{\phi\phi} = \tau_{rr} + Y(2-C)L(1)\left(\frac{b}{r}\right)^{2C}, \quad (11)$$

and $$p_i = Y\frac{(2-C)}{C}L(1)\left\{\left(\frac{b}{a}\right)^{2C} - 1\right\}, \quad \forall C \neq 0.$$

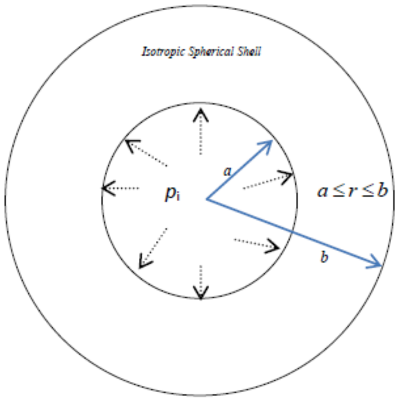

Figure 1. Spherical shell with applied internal pressure.

Non-dimensional components: we introduce the following non-dimensional components: $R = r/b$, $R_0 = a/b$, $\sigma_r = \tau_{rr}/Y$, $\sigma_\theta = \tau_{\theta\theta}/Y$, $P_0 = p_0/Y$, and $P_i = p_i/Y$. Equation (11) in non-dimensional form becomes:

$$\sigma_r = \frac{(2-C)}{C}L(1)\left\{1 - R^{-2C}\right\},$$

$$\sigma_\theta = \frac{(2-C)}{C}L(1)\left\{(1 - R^{-2C}) + CR^{-2C}\right\}, \quad (12)$$

and $$P_i = \frac{(2-C)}{C}L(1)\left\{R_0^{-2C} - 1\right\} \quad \forall C \neq 0.$$

Using the pressure condition, Eq.(12) becomes

$$\sigma_r = \frac{P_i(1-R^{-2C})}{(R_0^{-2c}-1)},$$

$$\sigma_\theta = \frac{P_i\left\{1-(1-C)R^{-2C}\right\}}{(R_0^{-2C}-1)} \quad \forall C \neq 0, \quad (13)$$

when $C = 0$, the stresses from Eq.(13) become

$$\sigma_r = \frac{-P_i R_0^{2C}\ln R}{R^{2C}\ln R}, \quad \sigma_\theta = \frac{-P_i\left\{2(1-C)R^{-2C}\ln R + R^{-2C}\right\}}{2R_0^{-2C}\ln R_0}$$

where: $$P_i = -L(1)\left\{2(2-C)R_0^{-2C}\ln R_0 + (R_0^{-2C}-1)\right\}. \quad (14)$$

Case 2: Pressure applied at the external surface

Suppose a uniform pressure be applied at the external outer surface of the spherical shell, say p_0, and the boundary conditions are $\tau_{rr} = 0$ at $r = a$ and $\tau_{rr} = -p$ at $r = b$ as shown in Fig. 2. Using boundary condition in Eq.(10), we get $A_1 = a^{2C}$. Now substituting the value of constant A_1 in Eq.(10) and using Eq.(6), we get

$$\sigma_r = \frac{P_0\left[1 - \left(\frac{R_0}{R}\right)^{2C}\right]}{(R_0^{2C}-1)}, \quad \sigma_\theta = \frac{P_0\left[1-(1-C)\left(\frac{R_0}{R}\right)^{2C}\right]}{(R_0^{2C}-1)} \quad \forall C \neq 0 \quad (15)$$

when $C = 0$; stresses from Eq.(15) become

$$\sigma_r = \frac{-P_{i0}(R_0/R)^{2C}\ln(R_0/R)}{R_0^{2C}\ln R},$$

$$\sigma_\theta = \frac{P_0\left\{(R_0/R)^{2C} - 2(1-C)(R_0/R)^{2C}\ln(R_0/R)\right\}}{2R_0^{2C}\ln R_0},$$

where: $$P_0 = L(1)\left\{2(2-C)R_0^{2C}\ln R_0 - R_0^{2C} + 1\right\}. \qquad (16)$$

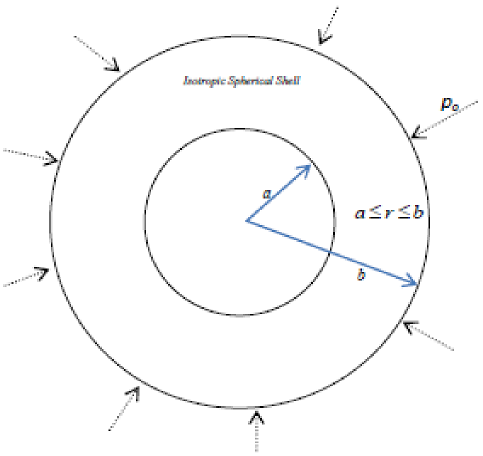

Figure 2. Spherical shell with applied external pressure.

The second equation from Eq.(11) is same Tresca yield condition, Hulsurkar /6/, when $L(1)$ = ½ and $C \to 0$ (fully plastic state) and when $L(1) = 1$ and $C \to 0$.

NUMERICAL ILLUSTRATION AND DISCUSSION

In Fig. 3, curves are drawn between pressure applied at the inner and outer boundary condition of the spherical shell and various radii ratios $R_0 = a/b$ for C = 0, 0.25, 0.75 and Lebesgue measure $L(1)$ = 0.5, 1, 2. It has been seen that spherical shell of compressible material requires a higher value of pressure at the centre of the spherical shell as compared to the incompressible material by using inner boundary condition, whereas at the outer boundary condition the shell of compressible material requires a higher value of pressure at the internal surface. The Lebesgue measure increases the pressure value at the inner and outer surface for compressible as well as incompressible materials.

In Figs. 4 and 5, curves are produced between stresses and radius ratio $R = a/b$ applied at the internal pressure and external pressure conditions and different values of Lebesgue measure $L(1)$ = 0.5, 1, 2. It has been observed that hoop stresses are maximum at the external surface for compressible material as compared to incompressible materials. The value of hoop stress increases with increased values of the Lebesgue measure.

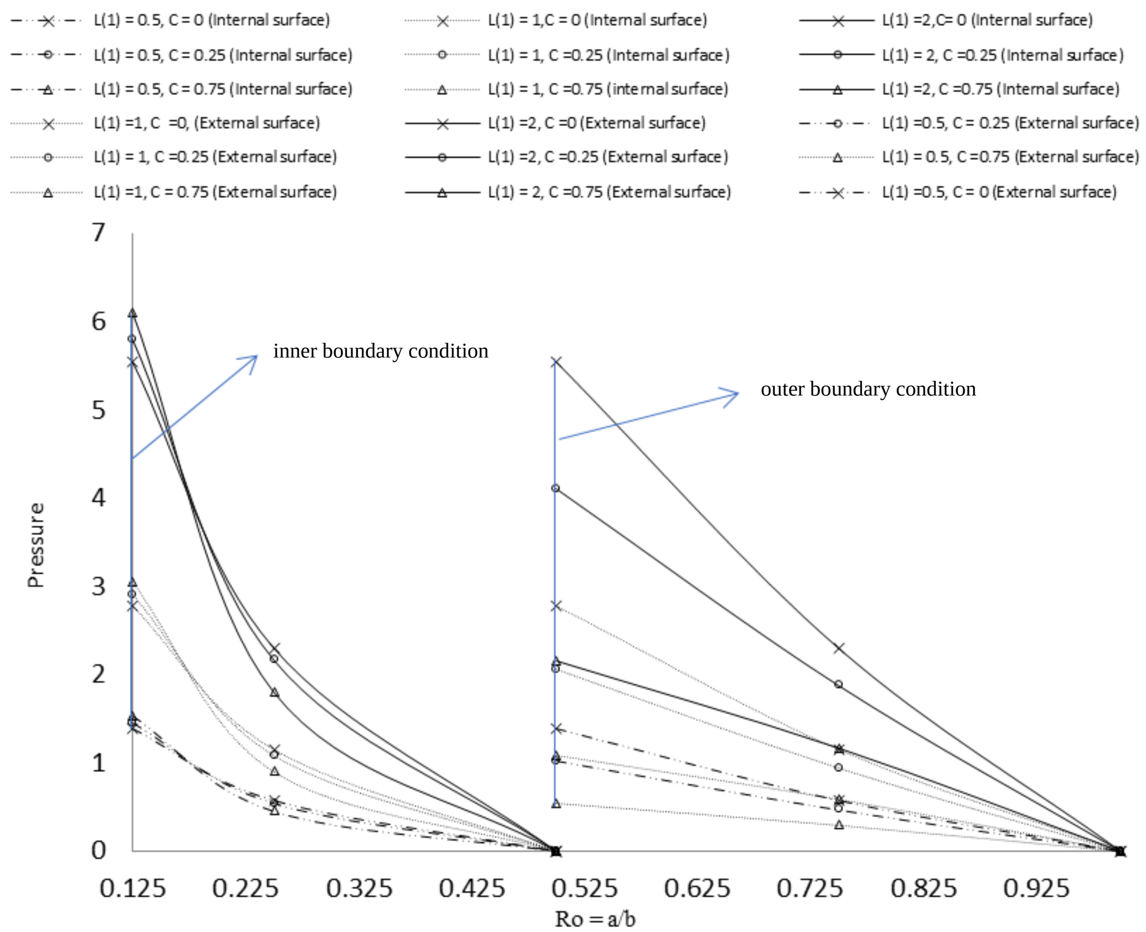

Figure 3. Graph between pressure and radii ratio ($R_0 = a/b$) for different values of Lebesgue measure $L(1)$ = 0.5, 1 and 2.

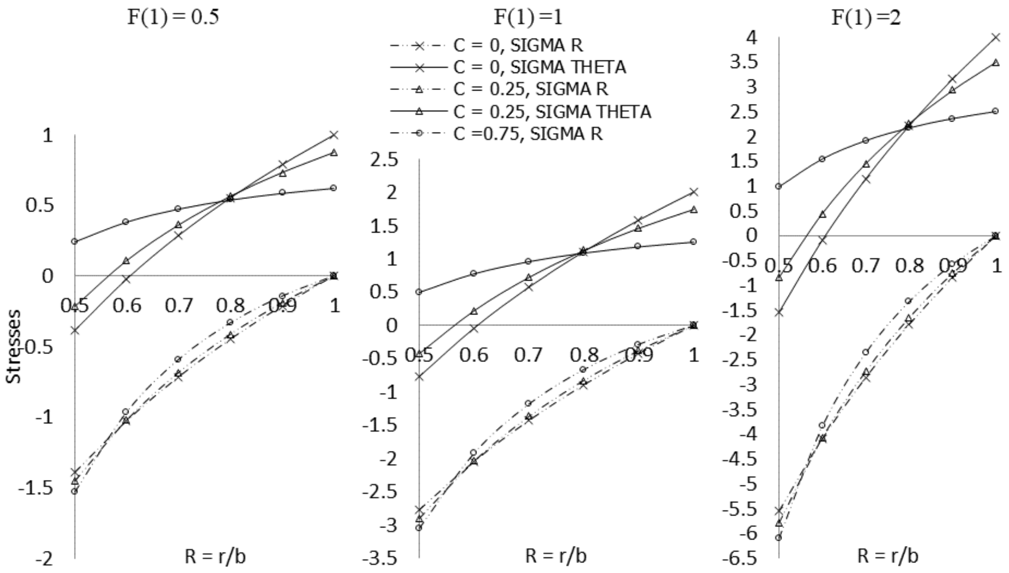

Figure 4. Graph between stresses and radii ratio $R = r/b$ applied at the internal pressure condition.

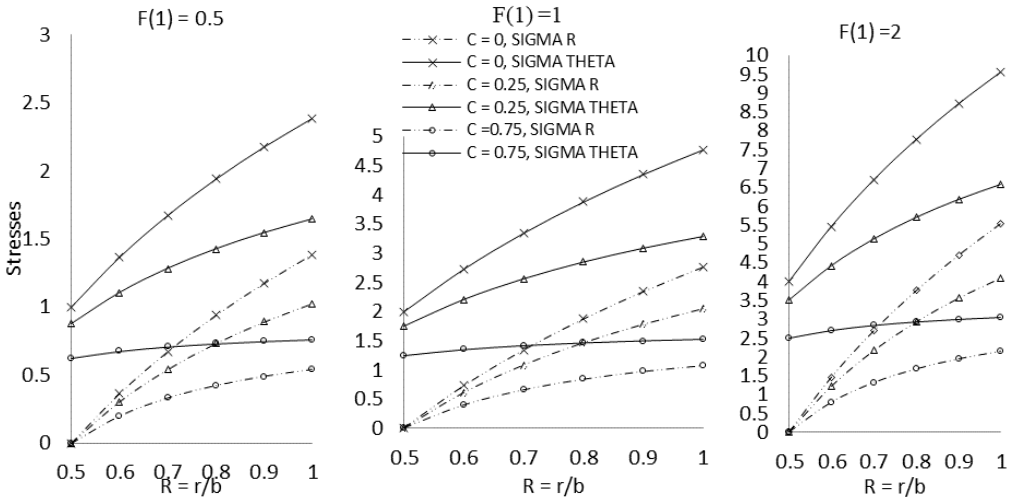

Figure 5. Graph between stresses and radii ratio $R = r/b$ applied at the external pressure condition.

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