ON THE ANALYSIS OF LENGTHWISE FRACTURE OF FUNCTIONALLY GRADED ROUND BARS

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Abstract

The strain energy release rate for lengthwise circular cylindrical cracks in round bars which are functionally graded in radial direction is derived by applying linear elastic fracture mechanics. The solution obtained is valid for a crack located arbitrarily in the radial direction of the bar cross-section. Besides, the modulus of elasticity and the shear modulus can be distributed arbitrarily in the radial direction. The functionally graded bars are loaded by axial forces, bending and torsion moments. The derived solution is applied to analyze the strain energy release rate for a lengthwise crack in a clamped functionally graded bar configuration. The external load of the clamped bar consists of an axial force and a torsion moment applied at the free end of the internal crack arm and a bending moment applied at the free end of the bar. In order to confirm the solution, the strain energy release rate in the clamped bar is determined also by considering the balance of energy and by applying the compliance method. The influence of various factors such as crack location in the radial direction, the material gradient and loading conditions on strain energy release rate in the clamped bar configuration are investigated and discussed.

INTRODUCTION

Functionally graded materials are inhomogeneous composites of two or more constituent materials. The most important feature of functionally graded materials is the smooth variation of their properties along one or more directions in the solid, /1-7/. Variation of material properties can be tailored in order to improve the performance of functionally graded structural members and components to externally applied loads. In recent years, functionally graded materials have been increasingly used as advanced structural materials in various engineering applications in aerospace, nuclear reactors, airplane industry and bioengineering.

Accessing of structural integrity, reliability and safety of functionally graded structural members and components is closely related with their fracture behaviour. Therefore, crack problems of functionally graded materials are an important subject of research that continues to attract the attention of academic community around the globe /8-12/.

Basic problems of fracture mechanics of functionally graded materials have been discussed in /8/. Methods for solving different crack problems in functionally graded materials have been developed. The fracture analyses performed and the results obtained can be useful for material scientists and design engineers which work in the field of functionally graded materials.

Various studies of fracture in functionally graded composite materials have been reviewed in /9/. Analyses of cracks oriented parallel or perpendicular to the direction of material gradient have been considered. Solutions for non-

Keywords

• functionally graded material
• lengthwise fracture
• round bar
• linear elastic behaviour

Keywords

• funkcionalni kompozitni materijal
• podužni lom
• okrugli štap
• linearno elastično ponašanje

Izvod

Primenom linearno elastične mehanike loma, izvedena je brzina oslobađanja deformacione energije kod podužnih kružnih cilindričnih prsliina u okruglim štapovima od funkcionalnog kompozitnog materijala radi slojevima u radijalnom pravcu. Dobijeno rešenje važi za prsljina preseka štapa. Osim toga, modal elastičnosti i modal klizanja su takođe proizvoljni u radijalnom pravcu. Izvedena rešenja su primjenjena u analizi brzine oslobađanja deformacione energije kod ukleštenog štapa od funkcionalnog kompozitnog materijala. Spoljašnja opterećenja ukleštenog štapa su aksijalna sila i moment. Radi provere rešenja, brzina oslobađanja deformacione energije deforme ukleštenog štapa se određuje razmatranjem ravnoteže energije i primenom metode popustljivosti. Razmotrni su i diskutovani uticaji pojedinih faktora, kao što su lokacija prsline u radijalnom pravcu, gradijen materijala i uslovi opterećenja na brzinu oslobađanja deformacione energije kod ukleštenog štapa.

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straight cracks have also been discussed. Investigations of fracture behaviour of functionally graded materials under static and cyclic fatigue crack loading conditions by applying linear elastic fracture mechanics have been summarized.

An engineering method for predicting the strength of functionally graded structural members containing cracks has been developed in /10/. The method has been applied to a functionally graded linear elastic beam subjected to three-point bending. The beam under consideration has a rectangular cross-section and is functionally graded in the thickness direction. A functionally graded linear elastic plate loaded in tension has also been analyzed.

The present paper is focused on deriving a solution to the strain energy release rate for a lengthwise circular cylindrical crack in functionally graded round bars exposed to external mechanical loading which induces axial forces and bending and torsion moments. It is assumed that the bars are functionally graded in radial direction. A solution to the strain energy release rate is obtained for arbitrary variation of the modulus of elasticity and the shear modulus within the radial coordinate. The solution obtained can be used for a cylindrical crack located arbitrary in radial direction. The solution is applied to analyse the strain energy release rate for a lengthwise cylindrical crack in a clamped functionally graded round bar. The strain energy release rate in the clamped bar is determined also by considering the balance of energy for verification. A further verification is performed by applying the compliance method. The influence of loading conditions, material properties and crack location on the strain energy release rate in the clamped bar is investigated and discussed.

**SOLUTION PROCEDURE TO THE STRAIN ENERGY RELEASE RATE**

In order to derive the strain energy release rate, a portion of a functionally graded round bar containing the crack front is considered (Fig. 1).

The bar cross-section is a circle of radius \( r_2 \). The axial force and torsion and bending moments in the bar cross-section ahead of the crack front are denoted by \( N_3 \), \( T_3 \) and \( M_3 \), respectively (Fig. 1). The lengthwise crack is a circular cylindrical surface of radius \( r_1 \). Thus, the crack front is a circle of radius \( r_1 \). In the present analysis, the internal crack arm is treated as a round bar of radius \( r_1 \). The external crack arm is treated as a bar of ring-shaped cross-section of internal radius \( r_1 \), and external radius \( r_2 \).

According to linear elastic fracture mechanics, the strain energy release rate, \( G \), for the crack problem in Fig. 1 is given as

\[
G = - \frac{\Delta U}{l_{cf} \Delta a},
\]

where: \( \Delta U \) is the change of strain energy; \( l_{cf} \) is the length of crack front; \( \Delta a \) is a small increase in crack length. The change of strain energy due to the increase of crack length is expressed as

\[
\Delta U = \Delta \eta \left[ \int_{0}^{2\pi} u_{01} r dr d\phi + \Delta \alpha \int_{0}^{2\pi} u_{02} r dr d\phi - \Delta \alpha \int_{0}^{2\pi} u_{03} r dr d\phi \right] (2)
\]

where the first and second terms in the right-hand side of the equation are the strain energies stored-in up in portions of length \( \Delta a \), in the two crack arms behind the crack front; the third term is the strain energy in the uncracked bar portion of length \( \Delta a \), ahead of the crack front; \( u_{01}, u_{02} \) and \( u_{03} \) are, respectively, strain energy densities in internal and external crack arms and in the uncracked bar portion ahead of the crack front.

The length of crack front is calculated as

\[
l_{cf} = 2\pi \eta. \quad (3)
\]

By substituting Eqs.(2) and (3) in Eq.(1), the strain energy release rate is expressed as

\[
G = \frac{1}{2\pi \eta} \left[ \int_{0}^{\eta} \int_{0}^{2\pi} u_{01} r dr d\phi + \int_{0}^{\eta} \int_{0}^{2\pi} u_{02} r dr d\phi - \int_{0}^{\eta} \int_{0}^{2\pi} u_{03} r dr d\phi \right]. \quad (4)
\]

The strain energy density in the internal crack arm is written as

\[
u_{01} = u_{01|\sigma} + u_{01|\tau}, \quad (5)
\]

where: \( u_{01|\sigma} \) and \( u_{01|\tau} \) are the densities of strain energy due to the axial force and bending moment and of the torsion moment, in respect. The strain energy density due to axial force and bending moment is written as

\[
u_{01|\sigma} = \frac{1}{2} \sigma \varepsilon, \quad (6)
\]

where: \( \sigma \) is the normal stress; \( \varepsilon \) is the strain. The normal stresses are obtained by Hooke’s law

\[
\sigma = E \varepsilon \quad, \quad (7)
\]

where: the modulus of elasticity varies continuously in the radial direction,

\[
E = E(r). \quad (8)
\]
According to Bernoulli’s hypothesis for plane sections, $\varepsilon$ is distributed linearly along the thickness of the internal crack arm

$$\varepsilon = \kappa_1 (z_1 - z_{n_0}) \ , \quad (9)$$

where

$$-\eta_1 \leq z_{1} \leq \eta_1 \ . \quad (10)$$

It should be noted that Bernoulli’s hypothesis is applicable since bars of a high length to thickness ratio are under consideration in the present paper. In Eq.(9), $\kappa_1$ is the curvature of the internal crack arm, $z_{n_0}$ is the coordinate of the neutral axis (Fig. 2).

It is obvious that the neutral axis shifts from the centroid since the internal crack arm is loaded by bending moment and axial force. By substituting Eqs.(7) and (9) in Eq.(6), one arrives at

$$u_{01\sigma} = \frac{1}{2} E \left[ \kappa_1 (z_1 - z_{n_0}) \right]^2 \ , \quad (11)$$

where: $E$ is a function of the radial coordinate.

The curvature and the neutral axis coordinate are determined by using the following equations for equilibrium of the cross-section of internal crack arm:

$$N_1 = \int \sigma \text{d}A \ , \quad (12)$$

$$M_1 = \int \sigma z_1 \text{d}A \ , \quad (13)$$

where: $A_1$ is the area of the internal crack arm cross-section; $N_1$ and $M_1$ are, respectively, the axial force and the bending moment in the cross-section of the internal crack arm behind the crack front. By using the designations in Fig. 2, Eqs.(12) and (13) are expressed in polar coordinates as

$$N_1 = \int_0^{\eta_1} \int_0^{2\pi} \sigma r^2 \text{d}r \text{d}\phi \ , \quad (14)$$

$$M_1 = \int_0^{\eta_1} \int_0^{2\pi} \sigma r^2 \sin \phi \text{d}r \text{d}\phi \ , \quad (15)$$

where: $\sigma$ is obtained by Eq.(7). In order to facilitate the integration in Eqs.(14) and (15), the distribution of lengthwise strain, Eq.(9), is rewritten as

$$\varepsilon = \kappa_1 (r \sin \phi - z_{n_0}) \ , \quad (16)$$

where: the polar angle, $\phi$, is defined in Fig. 2.

However, there are three unknowns: $\kappa_1$, $z_{n_0}$ and $M_1$, in Eqs.(14) and (15). Therefore, two other equations are worked-out by considering the equilibrium of the cross-section of external crack arm behind the crack front

$$N_2 = \int_{\eta}^{\pi - \eta} \int_0^{2\pi} \sigma_g r^2 \text{d}r \text{d}\phi \ , \quad (17)$$

$$M_2 = \int_{\eta}^{\pi - \eta} \int_0^{2\pi} \sigma_g r^2 \sin \phi \text{d}r \text{d}\phi \ , \quad (18)$$

where: $N_2$ and $M_2$ are, respectively, the axial force and the bending moment in the cross-section of external crack arm behind the crack front. The normal stress in the external crack arm, $\sigma_g$, is obtained by the Hooke’s law

$$\sigma_g = E \varepsilon_g \ , \quad (19)$$

where: lengthwise strain, $\varepsilon_g$, is distributed linearly along the height of the external crack arm cross-section,

$$\varepsilon_g = \kappa_2 (r \sin \phi - z_{2n_0}) \ , \quad (20)$$

In Eq.(20), $\kappa_2$ and $z_{2n_0}$ are the curvature and the coordinate on neutral axis, respectively. Since the axial force and the bending moment generate mode II crack loading conditions, the curvature of external crack arm is the same as the curvature of internal crack arm,

$$\kappa_2 = \kappa_1 \ . \quad (21)$$

Also, it is obvious that

$$M_1 + M_2 = M \ , \quad (22)$$

Equations (14), (15), (17), (18), (21) and (23) can be solved with respect to $\kappa_1$, $z_{n_0}$, $\kappa_2$, $z_{2n_0}$, $M_1$ and $M_2$ for arbitrary continuous variation of the modulus of elasticity with the radial coordinate. After that, $\kappa_1$ and $z_{n_0}$ are substituted in Eq.(11) to calculate $u_{01\sigma}$.

The strain energy density due to the torsion of internal crack arm is written as

$$u_{01\tau} = \frac{1}{2} \tau^2 \gamma \ , \quad (24)$$

where: $\tau$ and $\gamma$ are the shear stress and strain, respectively. The shear stress is obtained by applying the Hooke’s law

$$\tau = S \gamma \ , \quad (25)$$

where the shear modulus, $S$, varies arbitrary with the radial coordinate

$$S = S(r) \ . \quad (26)$$

Since we assume validity of the Bernoulli’s hypothesis, the shear strains are distributed linearly along the radius

$$\gamma = r \frac{\gamma}{}_{\eta} \ , \quad (27)$$

where:

$$0 \leq r \leq \eta_1 \ . \quad (28)$$

In Eq.(27), $\gamma$ is the shear strain at the periphery of the internal crack arm. The following equation for equilibrium of the internal crack arm cross-section is used to determine $\gamma$,

$$T_1 = \int \tau r \text{d}A \ , \quad (29)$$

where: $T_1$ is the torsion moment in the internal crack arm cross-section behind the crack front. Equation (29) is expressed in polar coordinates as

$$T_1 = \int_0^{\eta_1} \int_0^{2\pi} \tau r \text{d}r \text{d}\phi \ , \quad (30)$$

where: $\tau$ is obtained by Eq.(25). Equation (30) can be used to determine $\gamma$ for arbitrary continuous variation of the shear modulus with the radial coordinate.

By substituting Eqs.(25) and (27) in Eq.(24), one arrives at

$$u_{01\tau} = \frac{1}{2} S \left( \frac{\gamma}{}_{\eta} \right)^2 \ . \quad (31)$$
where: $S$ is a continuous function of the radial coordinate; $\gamma_{n}$ is obtained from Eq.(30).

The final expression for strain energy density in the internal crack arm cross-section behind the crack front is obtained by substituting Eqs.(11) and (31) in Eq.(5).

Equation (5) is applied also to calculate the strain energy density in the external crack arm cross-section behind the crack front. For this purpose, $z_{int}$ in Eq.(11) is replaced with $z_{ext}$. Also, $\gamma_{e}$ and $r_{1}$ in Eq.(31) are replaced with $\gamma_{e}$ and $r_{2}$, respectively. The shear strain at the periphery of the external crack arm, $\gamma_{e}$, is determined from the following equilibrium equation of the external crack arm cross-section:

$$T_{2} = \frac{r_{1} 2\pi}{n} \tau_{n} r^{2} \int d\phi, \quad (32)$$

where: $T_{2}$ is the torsion moment in the external crack arm behind the crack front. The shear stress in the external crack arm, $\tau_{e}$, is obtained by the Hooke’s law

$$\tau_{e} = \frac{S_{r} \gamma_{e}}{r_{2}^{2}}, \quad (33)$$

The strain energy density in the bar cross-section ahead of the crack front is obtained by Eq.(5). For this purpose, $\kappa_{1}$ and $z_{int}$ in Eq.(11) are replaced with $\kappa_{2}$ and $z_{ext}$, respectively ($\kappa_{2}$ and $z_{ext}$ are the curvature and the coordinate of neutral axis in the bar cross-section ahead of the crack front). Equations (14) and (15) are used to determine $\kappa_{2}$ and $z_{ext}$. For this purpose, $N_{1}, M_{1}, r_{1}$ and $\sigma$ are replaced with $N_{3}, M_{3}, r_{2}$ and $\sigma_{n}$, respectively. It is obvious that

$$N_{3} = N_{1} + N_{2}, \quad (34)$$

$$M_{3} = M_{1} + M_{2}, \quad (35)$$

The normal stress, $\sigma_{n}$, in the bar cross-section ahead of the crack front is found by Hooke’s law Eq.(7). The lengthwise strain is found by replacing $\kappa_{1}$ and $z_{int}$ in Eq.(9) with $\kappa_{2}$ and $z_{ext}$, respectively. In Eq.(31), $\gamma_{e}$ and $r_{1}$ are replaced with $\gamma_{e}$ and $r_{2}$, The shear strain, $\gamma_{e}$, at the periphery of the bar is determined by using Eq.(30). For this purpose, $T_{1}, r_{1}$ and $r$ are replaced with $T_{3}, r_{2}$ and $\gamma_{e}$, respectively. Obviously,

$$T_{3} = T_{1} + T_{2}, \quad (36)$$

The shear stress, $\tau_{n}$, is expressed by the Hooke’s law Eq.(25). The shear strain is obtained by replacing $\gamma_{n}$ and $r_{1}$ in Eq.(27) with $\gamma_{e}$ and $r_{2}$, respectively.

The strain energy release rate is calculated by substituting strain energy densities in Eq.(4). It should be mentioned that the solution to strain energy release rate derived in the present paper can be applied for functionally graded round bar configurations with arbitrary variation of modulus of elasticity and shear modulus with the radial coordinate.

**NUMERICAL EXAMPLE**

In the present section of the paper, the strain energy release rate for a lengthwise circular cylindrical crack in a clamped functionally graded round bar is analysed by applying the solution obtained in the previous section.

The round bar under consideration is depicted in Fig. 3. There is a lengthwise crack of length $a$ in the bar. The radius of bar cross-section is $r_{2}$. The internal crack arm has a circular cross-section of radius, $r_{1}$. The bar length is $l$. The bar is clamped at its right-hand end.

![Figure 3. Geometry and loading of a clamped round bar with a lengthwise cylindrical crack.](image)

External loading includes one bending moment, $M$, applied at the free end of the bar, one axial force, $F$, and one torsion moment, $T$, applied at the free end of internal crack arm. It is apparent that the internal crack arm is loaded by axial force and by torsion and bending moments, while the external crack arm is loaded in pure bending since $M$ is distributed on both crack arms. The uncracked bar portion, $a \leq x \leq l$, is loaded by axial force and by torsion and bending moments. Therefore,

$$N_{1} = F, \quad T_{1} = T, \quad (37)$$

$$N_{2} = 0, \quad T_{2} = 0, \quad (38)$$

$$N_{3} = F, \quad T_{3} = T. \quad (39)$$

The bending moments in the two crack arms, $M_{1}$ and $M_{2}$, are obtained from Eqs.(14), (15), (17), (18), (21) and (23).

For the functionally graded round bar shown in Fig. 3, it is assumed that the modulus of elasticity and the shear modulus vary continuously with the radial coordinate, according to the following power laws:

$$E = E_{0} + (E_{f} - E_{0}) \left(\frac{r}{r_{2}}\right)^{p}, \quad (40)$$

$$S = S_{0} + (S_{f} - S_{0}) \left(\frac{r}{r_{2}}\right)^{p}, \quad (41)$$

where:

$$0 \leq r \leq r_{2}. \quad (42)$$

In Eqs.(40) and (41), $E_{0}$ and $S_{0}$ are the values of modulus of elasticity and shear modulus at the centre of bar cross-section, respectively; $E_{f}$ and $S_{f}$ are values of the modulus of elasticity and shear modulus at the periphery of the bar, respectively. The material properties, $m$ and $p$, govern the distribution of $E$ and $S$ in the radial direction, respectively.

In order to calculate the strain energy density in the internal crack arm cross-section behind the crack front, first, $\kappa_{1}, z_{int}, \kappa_{2}, z_{ext}, M_{1}$ and $M_{2}$ are determined. For this purpose, by substituting Eq.(40) in Eqs.(14), (15), (17) and (18) and taking into account Eqs.(21) and (23), one obtains the following four equations with unknowns, $\kappa_{1}, z_{int}, \kappa_{2}, z_{ext}$ and $M_{1}$:
\[ N_1 = -E_0 \varepsilon_{zz} \eta \frac{1}{2} \pi k_1 + 2 \eta \varepsilon_{zz} \eta \frac{m+2}{m+4}, \]  
\[ M_1 = \frac{1}{4} E_0 \varepsilon_{zz} \eta \frac{m+4}{m+4}, \]  
\[ N_2 = -E_0 \varepsilon_{zz} \eta \frac{m+4}{m+4}, \]  
\[ M - M_1 = \frac{1}{4} E_0 \varepsilon_{zz} \eta \frac{m+4}{m+4}, \]

where: \( \eta = (E_0 - E_0)/r^2 \). Equations (43)-(46) are solved with respect to \( k_1, z_{1st}, z_{2nd} \) and \( M_1 \) by using the MatLab computer program. Then, \( u_{01,0} \) is derived by substituting Eq.(40), \( k_1 \) and \( z_{1st} \) in Eq.(11).

Further, by substituting Eqs.(25), (27) and (41) in (30), one arrives at

\[ T_1 = \frac{1}{2} \varepsilon_{zz} \eta \frac{m+4}{m+4}, \]

where: \( \varepsilon_{zz} = (S_1 - S_0)/r^2 \). From Eq.(47), one derives

\[ \gamma_d = 2T_1 (p + 4)/[S_0 \eta \frac{m+4}{m+4} + 4 \sigma \eta \frac{p+3}{p+4}]. \]

After that, \( u_{01,0} \) is determined by substituting Eq.(41) in Eq.(31). The strain energy density in the internal crack arm cross-section behind the crack front is found by substituting \( u_{01,0} \) and \( u_{02,2} \) in Eq.(5).

The strain energy density in the cross-section of external crack arm behind the crack front is determined in the following way. First, \( z_1 \) and \( z_{1st} \) are replaced, respectively, with \( z_2 \) and \( z_{2nd} \) in Eq.(11). Also, Eq.(40) is substituted in Eq.(11). Since the external crack arm is loaded in pure bending, \( u_{02} = u_{02,2} \).

Equations (43) and (44) are used to determine \( k_3 \) and \( z_{3nd} \). For this purpose, after replacing \( N_1, M_1, r_1, k_1 \) and \( z_{1st} \) with \( N_3, M_3, r_2, k_2 \) and \( z_{3nd} \), Eqs.(43) and (44) are solved with respect to \( k_3 \) and \( z_{3nd} \) by using the MatLab computer program. Then, \( u_{03,0} \) is obtained by replacing \( z_1, z_{1st} \) and \( k_1 \) with \( z_3, z_{3nd} \) and \( k_3 \) in Eq.(11). Equation (31) is applied to calculate \( u_{03,0} \). For this purpose, \( \gamma_d \) and \( r \) are replaced with \( \eta \) and \( r \), respectively. Equation (48) is used to determine \( \eta \). For this purpose, \( T_2 \), \( r_1 \) and \( \gamma_d \) are replaced, respectively, with \( T_3 \), \( r_2 \) and \( \gamma_d \). The strain energy density in the bar cross-section ahead of the crack front is written as \( u_{03} = u_{03,0} + u_{03,2r} \).

The strain energy release rate for the crack problem in Fig. 3 is derived by substituting \( u_{01}, u_{02} \) and \( u_{03} \) in Eq.(4). The result is

\[ G = \frac{1}{2 \pi R_1} (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5), \]

where

\[ \alpha_1 = \frac{k_1}{2} \left[ \frac{1}{4} E_0 \varepsilon_{zz} \eta \frac{m+4}{m+4} + 2 \eta \varepsilon_{zz} \eta \frac{m+4}{m+4} \right] \]

\[ \alpha_2 = \frac{\sigma}{\eta} \left[ \eta \frac{m+4}{m+4} + 2 \eta \varepsilon_{zz} \eta \frac{m+4}{m+4} \right] \]

In order to check Eq.(49), the strain energy release rate is obtained also by analysing the energy balance. For this purpose, a small increase of the crack length, \( \Delta a \), is assumed. The energy balance is written as

\[ F \delta u + M \delta \psi + T \delta \phi = \frac{\partial U}{\partial a} \delta a + Gl \delta a, \]

where: \( \psi \) and \( \phi \) are, respectively, increases in lengthwise displacement and angle of twist of the free end of the internal crack arm; \( \psi \) is the increase of the angle of rotation of free end of the bar; \( U \) is the strain energy cumulated in the bar. From Eq.(50), one obtains

\[ G = \frac{1}{2 \pi R_1} \left( \frac{\partial \delta u}{\partial a} + M \frac{\partial \delta \psi}{\partial a} + T \frac{\partial \delta \phi}{\partial a} - \frac{1}{2} \frac{\partial U}{\partial a} \right). \]

The integrals of Maxwell-Mohr are used to determine \( u, \psi \) and \( \phi \). The result is

\[ u = \frac{E_1 \alpha a + E_2 (l-a)}{\alpha_1}, \]

\[ \psi = \frac{E_1 \alpha a + E_2 (l-a)}{\alpha_2}, \]

\[ \phi = \frac{E_1 \alpha a + E_2 (l-a)}{\alpha_3}. \]

In Eq.(53), \( \alpha_1 \) and \( \alpha_2 \) are the lengthwise strains at the centres in cross-sections of internal crack arm and the uncracked beam portion, respectively. By substituting \( z_1 = 0 \) in Eq.(9), \( \alpha_1 \) is written as

\[ \alpha_1 = -\frac{k_1}{2} \frac{\varepsilon_{zz}}{m+4}. \]

Similarly, \( \alpha_2 \) is obtained as

\[ \alpha_2 = \frac{k_2}{2} \frac{\varepsilon_{zz}}{m+4}. \]

The strain energy cumulated in the bar is found by integrating strain energy densities in the volume of the internal and external crack arms and the uncracked bar portion

\[ U = a \int \left( \frac{\partial u}{\partial a} + M \frac{\partial \psi}{\partial a} + T \frac{\partial \phi}{\partial a} - \frac{1}{2} \frac{\partial U}{\partial a} \right) \frac{\partial u}{\partial a} \delta a. \]

By substituting Eqs.(53)-(58) in Eq.(52), one derives the following expression for the strain energy release rate:

\[ G = \frac{1}{2 \pi R_1} \left( -\frac{k_1}{2} \frac{\varepsilon_{zz}}{m+4} + M (k_1 - k_3) + \frac{T}{\eta} \frac{\gamma_d}{r} - \frac{\varepsilon_{zz}}{m+4} \right). \]
It should be noted that strain energy release rates calculated by Eq.(59) are exact match of these obtained by Eq.(49). This fact is a verification of strain energy release rate analyses developed in the present paper.

The compliance method is used also to verify the solution to the strain energy release rate Eq.(49). According to the compliance method, the strain energy release rate is written as

$$ G = \frac{1}{2\pi \eta} \left( \frac{f^2 dC_F}{da} + M^2 \frac{dC_M}{da} + T^2 \frac{dC_T}{da} \right), $$

where the compliances of the bar are expressed as

$$ C_F = \frac{u}{c}, \quad C_M = \frac{\eta}{M}, \quad C_T = \frac{\eta}{T}. $$

By substituting Eqs.(3), (53)-(55) in Eq.(60), one arrives at

$$ G = \frac{1}{4\pi \eta} \left[ F(-\kappa_1\zeta_1n_1 + \kappa_3\zeta_3m_3) + M(\kappa_1 - \kappa_3) + T \left( \frac{\gamma_1}{\eta} - \frac{\gamma_3}{\eta} \right) \right]. $$

Strain energy release rates obtained by Eq.(62) match exactly these calculated by Eq.(49) which is also a verification of the analysis developed in the present paper.

Parametric investigations are performed in order to evaluate the influence of crack location in radial direction, material properties and loading conditions on strain energy release rate for the crack problem shown in Fig. 3. For this purpose, calculations of strain energy release rate are carried out by Eq.(49). The results obtained are presented in non-dimensional form by using the formula $G_N = G/(E_0r_2)$. It is assumed that $l = 0.2$ m, $r_2 = 0.003$ m, $F = 500$ N, $M = 20$ Nm and $T = 30$ Nm.

The effect of crack location in the radial direction on strain energy release rate is analysed. For this purpose, $r_1/r_2$ ratio which characterizes the crack location in the radial direction is introduced. The strain energy release rate is calculated at three $r_1/r_2$ ratios. The results obtained are illustrated in Fig. 4 where the strain energy release rate in non-dimensional form is plotted against $E_1/E_0$ ratio at three $r_1/r_2$ ratios for $S_1/E_0 = 0.8$, $S_1/S_0 = 0.7$, $m = 0.4$ and $p = 0.5$. The curves in Fig. 4 indicate that the strain energy release rate decreases with increasing of $r_1/r_2$ ratio. Figure 4 shows also that increase of $E_1/E_0$ ratio leads to decrease in the strain energy release rate. This finding is attributed to the increase of the bar stiffness.

The influence of $S_1/E_0$ ratio and material property $m$, on strain energy release rate in the functionally graded bar shown in Fig. 3 is evaluated. For this purpose, calculations of strain energy release rate are performed at various $S_1/E_0$ ratios for three values of $m$. The strain energy release rate is plotted in non-dimensional form against $S_1/E_0$ ratio at $r_1/r_2 = 0.25$ in Fig. 5. One can observe in Fig. 5 that the strain energy release rate decreases with increasing of $S_1/E_0$ ratio. Increase of $m$ also leads to decrease of the strain energy release rate.

The effect of material property $p$, and $S_1/S_0$ ratio on the strain energy release rate is elucidated in Fig. 6 where the strain energy release rate in non-dimensional form is plotted against $S_1/S_0$ ratio at three values of $p$ for $r_1/r_2 = 0.25$. It can be observed that the strain energy release rate decreases with increasing of $S_1/S_0$ ratio (Fig. 6). The strain energy release rate decreases also with increasing of $p$.

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**Figure 4.** Strain energy release rate in non-dimensional form plotted against $E_1/E_0$ ratio (curve 1 at $r_1/r_2 = 0.25$; curve 2 at $r_1/r_2 = 0.50$; and curve 3 at $r_1/r_2 = 0.75$).

**Figure 5.** Strain energy release rate in non-dimensional form plotted against $S_1/E_0$ ratio (curve 1 at $m = 0.5$; curve 2 at $m = 0.7$; and curve 3 at $m = 0.9$).

**Figure 6.** Strain energy release rate in non-dimensional form plotted against $S_1/S_0$ ratio (curve 1 at $p = 0.4$, curve 2 at $p = 0.6$ and curve 3 at $p = 0.8$).
The influence of loading conditions on the strain energy release rate is investigated too. For this purpose, the strain energy release rate in non-dimensional form is plotted against $T/F$ ratio at three $T/M$ ratios in Fig. 7 for $r_1/r_2 = 0.25$. Figure 7 shows that the strain energy release rate increases with increasing of both $T/F$ and $T/M$ ratios.

Figure 7. Strain energy release rate in non-dimensional form plotted against $T/F$ ratio (curve 1 at $T/M = 0.5$; curve 2 at $T/M = 1.0$ and curve 3 at $T/M = 1.5$).

CONCLUSIONS

A solution procedure to the strain energy release rate for lengthwise cracks in functionally graded round bars is developed by applying methods of linear elastic fracture mechanics. The bars are loaded by axial forces and bending and torsion moments. The lengthwise cracks under consideration are circular cylindrical surfaces. The internal crack arm is treated as a bar of circular cross-section. The external crack arm is treated as a bar of ring-shaped cross-section. A solution to the strain energy release rate is derived assuming that the crack is located arbitrarily in radial direction. The solution holds for bars which are functionally graded in the radial direction (modulus of elasticity and shear modulus vary continuously in the radial direction).

The solution is applied to analyse the strain energy release rate for a lengthwise cylindrical crack in a clamped functionally graded round bar. Power laws are used to describe the distribution of the modulus of elasticity and the shear modulus in radial direction. The bar is loaded by an axial force and a torsion moment applied at the free end of the internal crack arm and a bending moment applied at the free end of the bar. The strain energy release rate in the clamped functionally graded round bar is derived also by considering the energy balance and by applying the compliance method for verification. Effects of the crack location in the radial direction, material gradients and the loading conditions on strain energy release rate are elucidated. It is found that the strain energy release rate decreases with increasing of the radius of internal crack arm cross-section. Material gradients in the radial direction are characterized by $E/E_0$ and $S/S_0$ ratios. Analysis reveals that the strain energy release rate decreases with increasing of $E/E_0$ and $S/S_0$ ratios. The increase of $m$ and $p$ leads also to decrease of strain energy release rate. The loading conditions are characterized by $T/F$ and $T/M$ ratios. The investigation shows that the strain energy release rate increases with increasing of $T/F$ and $T/M$ ratios.

The solution derived in the present paper can be applied to calculate the strain energy release rate when analysing lengthwise circular cylindrical cracks in round bars which are functionally graded in the radial direction. The results obtained may be useful in structural design of functionally graded round bars when considering their lengthwise fracture behaviour.

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