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ELASTIC-PLASTIC STRESS CONCENTRATIONS IN ORTHOTROPIC COMPOSITE SPHERICAL SHELLS SUBJECTED TO INTERNAL PRESSURE

KONCENTRACIJA ELASTOPLASTIČNIH NAPONA KOD ORTOTROPNIH KOMPOZITNIH SFERNIH LJUSKI OPTEREĆENIH UNUTRAŠNJIM PRITISKOM

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Keywords

- composites
- orthotropic materials
- transversely isotropic materials
- stresses
- Seth's transition theory
- spherical shell

Ključne reči

- kompoziti
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- transverzalni izotropni materijali
- naponi
- teorija prelaznih napona Seta
- sferna ljuska

Abstract

Elastic-plastic stress concentrations in spherical shells subjected to internal pressure are of much significance in the theory of structural components. The paper presents the study of elastic-plastic stress concentrations in orthotropic spherical shells made of composite materials subjected to internal pressure by using Seth's transition theory. It has been seen that the spherical shell of orthotropic composite material requires higher values of pressure at the inner surface as compared to shell of transversely isotropic material.

Izvod

Koncentracija elastoplastičnih napona kod sfernih ljuski opterećenih unutrašnjim pritiskom je od velikog značaja u teoriji nosećih konstrukcija. Predstavljeno je istraživanje koncentracije elastoplastičnih napona kod ortotropnih sfernih ljuski od kompozitnog materijala, opterećenih unutrašnjim pritiskom, primenom teorije prelaznih napona Seta. Poznato je da su potrebni veći pritisci na unutrašnjoj površini sferne ljuske od ortotropnog kompozitnog materijala u poređenju sa ljuskom od transverzalnog izotropnog materijala.

INTRODUCTION

Modelling the elastic-plastic behaviour of spherical and cylindrical shells and vessels under the influence of internal and external pressure has gained much significance in recent times. Shells made of materials with specific elastic behaviour are used in various mechanical components of satellites, submarines, hemispheric dome shaped antennas, automotives, helmets etc. Vessels made in the shape of shells when used to store fluids at high pressure, experience internal pressure, therefore it is necessary that the shell material retains its integrity under such conditions. Modelling spherical shells of isotropic materials is available in most standard textbooks /2, 3, 6, 8-10/. Miller /13/ evaluated solutions for stresses and displacements in a thick spherical shell subjected to internal and external pressure loads. You et al. /17/ presented a highly precise model to carry out elastic analysis of thick-walled spherical pressure vessels. The authors have studied the behaviour of shells particularly when some assumptions were made, such as: (i) incompressibility of the material used; (ii) creep strain law derived by Norton; (iii) yield condition of Tresca; and (iv) sssociated flow rules. The need for utilizing these specially appointed semi-experimental laws in elastic-plastic transition depends on the approach that the transition is a linear phenomenon which is unrealistic. Deformation fields related to irreversible phenomenon, such as elastic-plastic disfigurements, creep relaxation, fatigue and crack etc. are nonlinear in character. The traditional measures of deformation are not adequate to manage transitions. The concept of generalized strain measures and transition theory given by /4/ has been applied to find elastic-plastic stresses in various problems by solving non-linear differential equations at the transition points. Thakur /22/ successfully analysed creep transition stresses of a thick isotropic spherical shell by finitesimal deformation under steady state of temperature and internal pressure by using Seth's transition theory. All these problems, based on the recognition of the transition state as separate state, necessitates showing the existence of the used constitutive equation for that state. In this paper we have studied the behaviour of orthotropic composite material when modelled in the shape of a spherical shell subjected to internal pressure. Two types of composite materials exhibiting orthotropic elastic behaviour are used to check the elastic-plastic stress concentrations. The first material is a boron-aluminium fibre reinforced composite, composed of uniaxial boron fibres in a matrix of 6061 aluminium alloy, tested for the stiffness matrix constants by acoustic resonance spectroscopy, /12/. The second material is a graphite-magnesium fibre reinforced composite, composed of continuous uniaxial graphite fibres in a magnesium matrix, studied for orthotropic elastic constants, /11/. Observations of these two orthotropic composites are compared to observations of transversely isotropic materials, i.e. titanium, whose stiffness constants are given in /14/ and barium-titanate, given in /15/.

GOVERNING EQUATIONS

We consider a spherical shell of constant thickness with internal and external radii a and b, respectively, under internal pressure p_i.

Displacement coordinates: the components of displacement in spherical coordinates are taken as:

$$u = r(1-\beta),\ v = 0,\ w = 0\,, \tag{1}$$

where: β is position function depending on r. The generalized components of strain are given by /5-7/ as:

$$e_{rr} = \frac{1}{n}\left[1-(r\beta'+\beta)^n\right],\quad e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right] = e_{\phi\phi}, \quad e_{r\theta} = e_{\theta\phi} = e_{\phi r} = 0. \tag{2}$$

where: n is the measure; and $\beta' = d\beta/dr$.

Stress-strain relation: the stress-strain relations for the isotropic material are given by /1/:

$$T_{ij} = c_{ijkl} e_{kl}\ , \quad (i, j, k, l = 1,2,3)$$

where: T_{ij} and e_{kl} are the stress and strain tensors, respectively. These nine equations contain a total of 81 coefficients e_{ijkl}, but not all coefficients are independent. The symmetry of T_{ij} and e_{ij} reduces the number of independent coefficients to 36. For elastic orthotropic materials which have three mutually orthogonal planes of elastic symmetry, these independent coefficients reduce to 12 and to 9 if the coefficients are symmetric. The constitutive equations for orthotropic media are given by /16/:

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{31} \\ e_{12} \end{bmatrix}. \tag{3}$$

Substituting Eq.(2) in Eq.(3), we get:

$$T_{rr} = \frac{c_{11}}{n}\left[1-(r\beta'+\beta)^n\right] + \frac{1}{n}(c_{12}+c_{13})(1-\beta^n)\,,$$
$$T_{\theta\theta} = \frac{c_{21}}{n}\left[1-(r\beta'+\beta)^n\right] + \frac{1}{n}(c_{22}+c_{23})(1-\beta^n)\,,$$
$$T_{\phi\phi} = \frac{c_{31}}{n}\left[1-(r\beta'+\beta)^n\right] + \frac{1}{n}(c_{32}+c_{33})(1-\beta^n)\,,$$
$$T_{r\theta} = T_{\theta\phi} = T_{\phi r} = 0\,. \tag{4}$$

Equation of equilibrium: the equations of equilibrium are:

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta} + \frac{2T_{rr} - T_{\theta\theta} - T_{\phi\phi} + T_{r\theta}\cot\theta}{r} = 0\,,$$
$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{3T_{r\theta} + (T_{\theta\theta} - T_{\phi\phi})\cot\theta}{r} = 0\,,$$
$$\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{1}{r}\frac{\partial T_{\phi\theta}}{\partial\theta} + \frac{3T_{r\phi} + 2T_{\theta\theta}\cot\theta}{r} = 0\ . \tag{5}$$

Substituting Eq.(4) in Eq.(5), we see that the equations of equilibrium are all satisfied except:

$$\frac{\partial T_{rr}}{\partial r} + \frac{2T_{rr} - T_{\theta\theta} - T_{\phi\phi}}{r} = 0\,, \tag{6}$$

and

$$\frac{T_{\theta\theta} - T_{\phi\phi}}{r}\cot\theta = 0\ . \tag{7}$$

From Eq.(7), the only case of interest is

$$T_{\phi\phi} - T_{\theta\theta} = 0\ . \tag{8}$$

Equation (8) is satisfied by $T_{\theta\theta}$ and $T_{\phi\phi}$ as given by Eq.(2). If $c_{21} = c_{31}$, $c_{22} - c_{33} = c_{32} - c_{23}$, the equation of equilibrium from Eq.(6) becomes:

$$\frac{\partial T_{rr}}{\partial r} + \frac{2(T_{rr} - T_{\theta\theta})}{r} = 0\ . \tag{9}$$

Critical points: by substituting Eq.(4) into Eq.(9), we get a nonlinear differential equation with respect to β:

$$\beta\frac{dP}{d\beta} = \frac{2(c_{11}-c_{21})\left\{1-\beta^n(P+1)\right\}}{nc_{11}\beta^n P(P+1)^{n-1}} - \frac{(c_{12}+c_{13})(P+1)^{1-n}}{c_{11}} - -(P+1) + \frac{2[c_{12}+c_{13}-(c_{22}+c_{23})](1-\beta^n)}{nc_{11}\beta^n P(P+1)^{n-1}} \tag{10}$$

where: P is function of β; and β is function of r only.

Transition points: the transition points of β in Eq.(10) are $P = 0$, $P \to -1$, and $P \to \pm\infty$.

Boundary condition: the boundary conditions of the problem are given by:

$$r = a,\ \ \tau_{rr} = -p_i\ ,\ \text{ and }\ r = b,\ \ \tau_{rr} = 0. \tag{11}$$

PROBLEM SOLUTION

For finding the elastic-plastic stresses, the transition function is taken through the principal stresses (see /5-7, 18-36/) at transition point $P \to \pm\infty$. We define the transition function ζ as:

$$\zeta = 1 - \frac{nT_{rr}}{(c_{11}+c_{12}+c_{13})} \cong \frac{1}{(c_{11}+c_{12}+c_{13})} \times \times\left[\beta^n(P+1)^n c_{11} + (c_{12}+c_{13})\beta^n\right] \tag{12}$$

where: ζ be the transition function of r only. Taking the logarithmic differentiation of Eq.(12) with respect to r and using Eq.(10), we get

$$\frac{d(\log\zeta)}{dr} = \frac{2}{r\beta^n} \times \times\frac{\left[1-\beta^n(P+1)^n\right](c_{11}-c_{21}) + (1-\beta^n)\left[c_{12}+c_{13}-(c_{22}+c_{23})\right]}{c_{11}(P+1)^n + (c_{12}+c_{13})} \tag{13}$$

Taking the asymptotic value of Eq.(13) as $P \to \pm\infty$ and integrating, we get:

$$R = A_1 r^{-2K}\ , \tag{14}$$

where: A_1 is a constant of integration; and $K = c_{11} - c_{21}/c_{11}$. From Eq.(12) and Eq.(14), we have

$$T_{rr} = \frac{(c_{11}+c_{12}+c_{13})}{n}\left[1 - A_1 r^{-2K}\right]. \tag{15}$$

Using the boundary condition from Eq.(11) into Eq.(15), we get

$$A_1 = b^{2K} \quad \text{and} \quad p_i = -\frac{(c_{11}+c_{12}+c_{13})}{n}\left[1-\left(\frac{b}{a}\right)^{2K}\right]. \qquad (16)$$

Substituting Eq.(15) into Eq.(6) and using Eq.(16) and Eq.(8), we get

$$T_{\theta\theta} - T_{rr} = \frac{(c_{11}+c_{12}+c_{13})K}{n}\left(\frac{b}{r}\right)^{2K}. \qquad (17)$$

Initial yielding: from Eq.(17), it is seen that $|T_{\theta\theta} - T_{rr}|$ is maximal at the inner surface (that is at $r = a$), therefore yielding of the shell will take place at the inner surface of the shell and Eq.(17) can be written as:

$$\left|T_{\theta\theta} - T_{rr}\right|_{r=a(\text{inner surface})} = \left|\frac{(c_{11}+c_{12}+c_{13})K}{n}\left(\frac{b}{a}\right)^{2K}\right| \equiv Y(\text{yielding}) \qquad (18)$$

Using Eq.(18) in Eqs.(15)-(17), we get the orthotropic transitional stresses as in non-dimensional components as:

$$\sigma_{rr} = -\frac{P_i(1-R^{-2K})}{1-R_0^{-2K}},$$

$$\sigma_{\theta\theta} = -\frac{P_i(1-R^{-2K})}{1-R_0^{-2K}} + R^{-2K}, \quad \text{and} \qquad (19)$$

$$P_i = -K^{-1}R_0^{2K}(1-R_0^{-2K}),$$

where: $R = r/b$; $R_0 = a/b$; $\sigma_{rr} = T_{rr}/Y$; $\sigma_{\theta\theta} = T_{\theta\theta}/Y$; and $P_i = p_i/Y$.

Fully-plastic state: for the fully-plastic case /5/, $c_{11} = c_{13} = -c_{12}$, $c_{23} = c_{21} = -c_{22}$, the stresses and pressure from Eq.(19) become:

$$\sigma_{rr} = -\frac{P_f(1-R^{-2K_1})}{1-R_0^{-2K_1}},$$

$$\sigma_{\theta\theta} = -\frac{P_f(1-R^{-2K_1})}{1-R_0^{-2K_1}} + R^{-2K_1}, \quad \text{and} \qquad (20)$$

$$P_f = -K_1^{-1}R_0^{2K_1}(1-R_0^{-2K_1}).$$

where: $R = r/b$; $R_0 = a/b$; $\sigma_{rr} = T_{rr}/Y^*$; $\sigma_{\theta\theta} = T_{\theta\theta}/Y^*$; $K_1 = (c_{12} - c_{22})/c_{12}$; and $P_f = p_f/Y^*$.

NUMERICAL RESULTS AND DISCUSSIONS

The above investigations elaborate the initial yielding and fully plastic state of spherical shells subjected to internal pressure. The cases of two shells made of orthotropic fibre-reinforced composites (i.e. boron-aluminium composite and graphite-magnesium composite) and transversely isotropic material (i.e. titanium and barium-titanate) are considered. In Fig. 1, curves are drawn between pressure and radii ratio $R_0 = a/b$ at initial yielding state by using Table 1. It has been seen that the spherical shell made of orthotropic composite material requires higher values of pressure at the inner surface as compared to the spherical shell made of transversely isotropic materials.

Curves are produced in Figs. 2-3, between stresses and radii ratio $R = r/b$ for elastic-plastic transition state and fully plastic state. It has been observed that the circumferential stress has a maximal value at the inner surface of the shell made of the orthotropic material (i.e. boron-aluminium composite and graphite-magnesium composite) as compared to the shell disc made of the transversely isotropic material (i.e. titanium and barium-titanate), but reverse results in the case of full plasticity.

Table 1. Elastic stiffness constants.

Material symmetry	Materials	Elastic stiffness constants in units of GPa								
		C_{11}	C_{22}	C_{33}	C_{44}	C_{55}	C_{66}	C_{12}	C_{13}	C_{23}
Orthotropic*	boron-aluminium composite	185.9	183.5	246.1	55.1	55.8	50.8	74.9	60.3	59.4
	graphite-magnesium composite	28.19	27.08	174.68	17.91	17.70	8.76	10.66	12.41	12.41
Transversely isotropic**	titanium	162.4	180.7	180.7	46.7	-	-	92.0	69.0	-
	barium-titanate	168	189	189	5.46	-	-	78	71	-

* Orthotropic materials have 9 independent material parameters.

** Transversely isotropic materials have 5 independent material parameters.

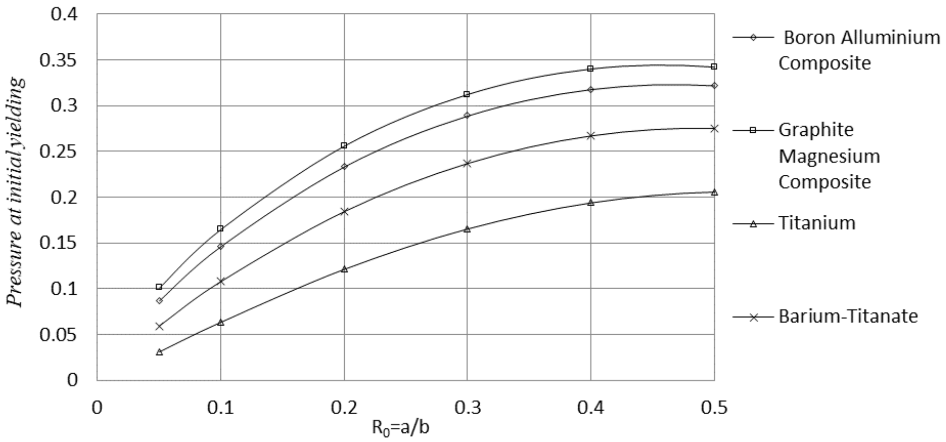

Figure 1. Effect of pressure in shells along the radius $R_0 = a/b$ at initial yielding.

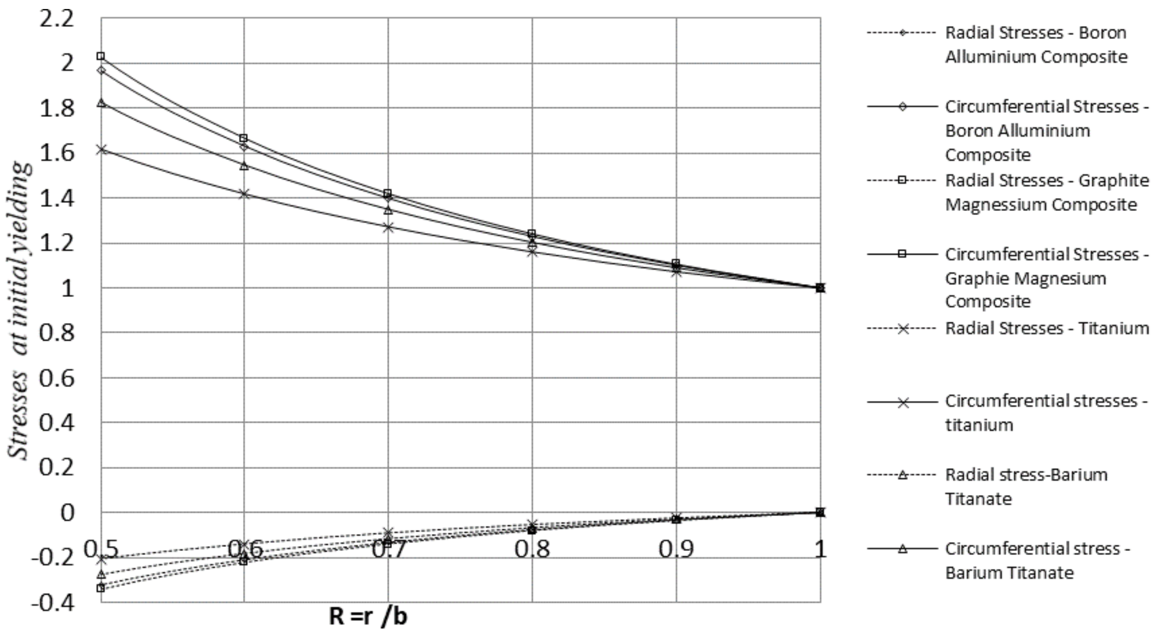

Figure 2. Effect of stresses in shells along the radius $R = r/b$ at elastic-plastic state.

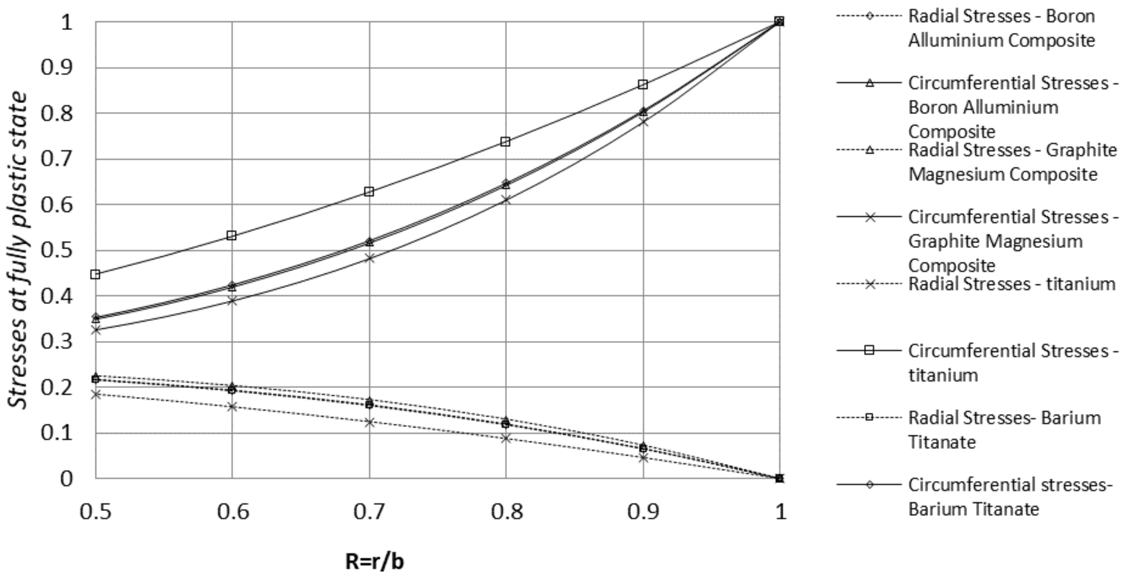

Figure 3. Effect of stresses in spherical shells along the radius $R = r/b$ for fully-plastic state.

REFERENCES

1. Sokolnikoff, I.S., Mathematical Theory of Elasticity, 2nd Ed., McGraw-Hill Inc., New York, 1953.
2. Timoshenko, S.P., Woinowsky-Krieger, S., Theory of Plates and Shells, 2nd Ed., McGraw-Hill Inc., New York, 1959.
3. Lubhan, J.D., Felgar, R.P., Plasticity and Creep of Metals, Wiley, New York, USA, 1961.
4. Seth, B.R. (1962), *Transition theory of elastic-plastic deformation, creep and relaxation*, Nature, 195: 896-897. doi:10.1038/195896a0.
5. Seth, B.R. (1963), *Elastic-plastic transition in shells and tubes under pressure*. ZAMM, 43(7-8): 345-351. doi: 10.1002/zamm.19630430706
6. Fung, Y.C., Foundations of Solid Mechanics, Englewood Cliffs, N.J. Prentice-Hall, 1965.
7. Seth, B.R. (1966), *Measure concept in mechanics,* Int. J Non-linear Mech., 1(1): 35-40.
8. Parkus, H., Thermoelasticity, Springer-Verlag, Wien, 1976. doi: 10.1007/978-3-7091-8447-9
9. Kraus, H., Creep Analysis, John Wiley & Sons Inc., New York, USA, 1980.
10. Boyle, J.T., Spence, J., Stress Analysis for Creep, Butterworths-Heinemann, London, 1983.
11. Datta, S., Ledbetter, H. (1989), *Graphite-magnesium elastic constants: composite and fiber*, in Höller P., Hauk V., Dobmann G., Ruud C.O., Green R.E. (eds) Nondestructive Characterization of Materials. Springer, Berlin, Heidelberg. doi: 10.1007/978-3-642-84003-6_10
12. Ledbetter, H., Fortunko, C., Heyliger, P. (1995), *Orthotropic elastic constants of a boron-aluminum fiber-reinforced composite: An acoustic-resonance-spectroscopy study*, J Appl. Physics 78: 1542-1546. doi: 10.1063/1.360247
13. Miller, G.K. (1995), *Stresses in a spherical pressure vessel undergoing creep and dimensional changes*, Int. J Solids & Struc., 32(14):2077-2093.
14. Uyaner, M., Akdemir, A., Erim, S., Avci, A. (2000), *Plastic zones in a transversely isotropic solid cylinder containing a ring-shaped crack*, Int. J Fracture, 106(2):161-175. doi: 10.1023/A:1007689718605

15. Royer, D., Dieulesaint, E., Elastic Waves in Solids I, Springer-Verlag, Berlin Heidelberg, 2000.
16. Altenbach, H., Altenbach, J., Kissing W., Mechanics of Composite Structural Elements, Springer-Verlag Berlin Heidelberg, 2004. doi: 10.1007/978-3-662-08589-9
17. You, L.H., Zhang J.J., You X.Y. (2005), *Elastic analysis of internally pressurized thick-walled spherical pressure vessels of functionally graded materials*, Int. J Press. Ves. & Piping, 82 (5):347-354. doi: 10.1016/j.ijpvp.2004.11.001
18. Gupta, S.K., Thakur, P. (2007), *Creep transition in a thin rotating disc with rigid inclusion*, Defence Sci. J, 57(2):185-195.
19. Gupta, S.K., Thakur, P. (2008), *Creep transition in an isotropic disc having variable thickness subjected to internal pressure*, Proc. Nat. Acad. of Science, India, Section-A, Vol.78, Part-1, pp.57-66.
20. Thakur, P. (2009), *Elastic-plastic transition stresses in a transversely isotropic thick-walled cylinder subjected to internal pressure and steady-state temperature*, Thermal Sci. 13(4): 107 -118. doi: 10.2298/TSCI0904107P
21. Thakur P. (2011), *Elastic-plastic transition stresses in rotating cylinder by finite deformation under steady-state temperature*, Therm. Sci. 15(2): 537-543.
22. Thakur, P. (2011), *Creep transition stresses of a thick isotropic spherical shell by finitesimal deformation under steady state of temperature and internal pressure*, Therm. Sci. 15(Suppl.2): S157-S165.
23. Thakur, P. (2014), *Steady thermal stress and strain rates in a rotating circular cylinder under steady state temperature*, Therm. Sci. 18 (Suppl. 1): S93-S106. doi: 10.2298/TSCI11031 8079P
24. Thakur, P. (2014), *Steady thermal stress and strain rates in a circular cylinder with non-homogeneous compressibility subjected to thermal load*, Thermal Sci. 18 (Supp. 1): S81- S92. doi: 10. 2298/TSCI110315080P
25. Thakur, P. (2015), *Analysis of thermal creep stresses in transversely thick-walled cylinder subjected to pressure*, Struc. Integ. and Life, 15(1): 19-26.
26. Thakur, P., Singh, S.B., Lozanović Šajić, J. (2015), *Thermo elastic-plastic deformation in a solid disk with heat generation subjected to pressure*, Struc. Integ. and Life, 15(3):135-142.
27. Thakur, P., Kumar, S., Singh, J., Singh, S.B. (2016), *Effect of density variation parameter in a solid disk*, Struc. Integ. and Life, 16(3): 143-148.
28. Thakur, P., Kumar, S. (2016), *Stress evaluation in a transversely isotropic circular disk with an inclusion*, Struc. Integ. and Life, 16(3): 155-160.
29. Gupta, N., Thakur, P., Singh, S.B. (2016), *Mathematical method to determine thermal strain rates and displacement in a thick-walled spherical shell*, Struc. Integ. and Life 16(2): 99-104.
30. Thakur, P., Kaur, J., Singh, S.B. (2016), *Thermal creep transition stresses and strain rates in a circular disc with shaft having variable density*, Eng. Comput., 33(3): 698-712. doi: 10.1108/EC-05-2015-0110
31. Thakur, P., Gupta, N., Singh, S.B. (2017), *Creep strain rates analysis in cylinder under temperature gradient materials by using Seth's theory*, Eng. Comput., 34(3): 1020-1030. doi: 10. 1108/EC-05-2016-0159
32. Thakur, P., Pathania, D., Verma, G., Singh, S.B. (2017), *Elastic-plastic stress analysis in a spherical shell under internal pressure and steady state temperature*, Struc. Integ. and Life, 17(1): 39-43.
33.Thakur, P., Singh, S.B., Pathania, D., Verma, G. (2017), *Thermal creep stress and strain analysis in a non-homogeneous spherical shell*, J Theor. and Appl. Mech., Warsaw, 55(4): 1155-1165. doi: 10.15632/jtam-pl.55.4.1155
34. Thakur, P., Mahajan, P., Kumar, S. (2018), Creep stresses and strain rates for a transversely isotropic disc having the variable thickness under internal pressure, Struc. Integ. and Life, 18(1): 15-21.
35. Thakur, P., Shahi S., Singh S.B., Emmanuel F.S. (2018), *Exact solution of rotating disc with shaft problem in the elastoplastic state of stress having variable density and thickness*, Struc. Integ. and Life, 18(2): 128-134.
36. Thakur, P., et al. (2018), *Modelling of creep behaviour of a rotating disc in the presence of load and variable thickness by using Seth transition theory*, Struc. Integ. and Life, 18(2): 135-142.