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ELASTIC-PLASTIC STRESS ANALYSIS OF SPHERICAL SHELL UNDER INTERNAL AND EXTERNAL PRESSURE

ELASTOPLASTIČNA ANALIZA NAPONA U SFERNOJ LJUSCI PRI DEJSTVU UNUTRAŠNJEG I SPOLJAŠNJEG PRITISKA

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Ključne reči
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Abstract

A thick-walled spherical shell made up of homogeneous material subjected to combined effect of internal and external pressure has been analysed. The objective of this paper is to provide guidance in designing the spherical shell so that collapse of spherical shell due to internal and external pressure can be avoided. The problem is based on elastic-plastic transition phenomenon and the solution has been obtained by using the concept of generalized strain measures and Seth's transition theory. The transition theory does not assume adhoc assumptions like incompressibility and yield conditions. The radial and circumferential stresses have been evaluated at the internal surface of the spherical shell for compressible as well as incompressible materials. It has been observed that the spherical shell of incompressible material requires a high pressure to start initial yielding in the shell as compared to the spherical shell of a compressible material. The results are derived numerically and shown graphically.

Izvod

Analizirana je debelozidna sferna ljuska od homogenog materijala, opterećena kombinovanom dejstvu unutrašnjeg i spoljašnjeg pritiska. Cilj ovog rada je davanje preporuka u projektovanju sferne ljuske, tako da može izbeći njen kolaps usled unutrašnjeg i spoljašnjeg pritiska. Problem se zasniva na elastoplastičnom prelaznom fenomenu, a rešenje je dobijeno primenom koncepta generalisanih mera deformacije i teorije prelaznih napona Seta. Teorija prelaznih napona ne podrazumeva 'adhoc' pretpostavke, kao što su nestišljivost i uslove tečenja. Radijalni i obimski naponi su izračunati za unutrašnju površinu sferne ljuske za stišljiv, kao i za nestišljiv materijal. U slučaju sferne ljuske od nestišljivog materijala, uočava se da je potreban viši pritisak za inicijalno tečenje ljuske, u poređenju sa sfernom ljuskom od stišljivog materijala. Rezultati su dobijeni numeričkim postupkom i prikazani su grafički.

INTRODUCTION

Engineers have discovered different utilizations of spherical shell structures in aviation, chemical, civil and mechanical ventures, for example, in rapid centrifugal separators, gas turbines for high-control flying machine motors, turning satellite structures, certain rotor frameworks and pivoting magnetic shields, /1/. To expand the quality of spherical shells, it is in this way an imperative for architects to concentrate the stress investigation in spherical shells under different conditions. The issues of a homogeneous and an isotropic spherical shell under inner pressure have been found in a large portion of the standard elasticity and plasticity books. Nowadays, pressurized (inner and outer) shells have turned scientists into a state of enthusiasm, because of their wide application in industry. A pressurized spherical storage vessel is an example of the problem of spherical shell under internal and external pressure, which is useful for storing high pressure fluids. Derrington et al. /1/ have discussed the effect of temperature on the thick spherical shell subjected to internal pressure. Results are derived according to the regions in which yield first occurs. It has been observed that yielding can be occur at any given radius or at any position in the spherical shell by making appropriate choice of pressure and temperature. Shima et al. /2/ presented mathematical model based on small deflection thin shell theory in which the problem of the spherical shell under external pressure is solved under conjunction with variational principles. Cong et al. /3/ discussed the nonlinear axisymmetric response of shallow spherical FGM shells under mechanical, thermal loads and different boundary conditions based on classical theory of shells. Sharma et al. /4-6/ worked on the problem of functionally graded cylinder under the effect of internal and external pressure. The results for elastic-plastic and creep stress distributions are obtained by using the idea of Lebesgue strain measure. The Lebesgue strain measure is helpful to solve such type of problems by neglecting the various assumptions in classical mechanics that is: (i) incompressibility of the material, (ii) creep strain laws derived by Norton, (iii) yield condition like that of Tresca, (iv) associated flow rule. The need of utilization of these specially appointed semi-experimental laws in established hypothesis of elastic-plastic transition depends on the approach that the transition is a linear phenomenon which is unrealistic. Under elastic plastic and creep curves, the structure of the object experiences changes and revamps them to bring about nonlinear impacts. In this way, it proposes that at transition conduct, nonlinear terms are huge and cannot be disregarded. Deformation fields related with irreversible phenomenon, for example, elastic plastic disfigurements, creep relaxation, fatigue and crack, and so on, are nonlinear in character. The traditional measures of deformation are not adequate to manage transitions and henceforth the relating constitutive conditions of the medium are entangled. There emerges the need of speculation of strain rate measure with the goal that they can be utilized as a part of the concept of generalized strain measures and the transition theory has been applied to find elastic-plastic stresses in various problems by solving nonlinear differential equations at transition points; for example Thakur et al. /7-10/ analysed elastic-plastic and creep transition in the spherical shell, cylinder and disc with various conditions. All these problems based on the recognition of the transition state as a separate state necessitates showing the existence of the constitutive equation for that state. In this paper, we have determined the outcomes for effective pressure required to begin initial yielding in the spherical shell. The stresses under pressure in the spherical shell are figured for compressible and additionally for incompressible materials. The outcomes acquired are demonstrated graphically.

FORMULATION OF MATHEMATICAL PROBLEM

We consider here a thick-walled spherical shell, with internal and external radii a and b, respectively, that is subjected to uniform internal and external pressures p_1 and p_0 respectively. It is convenient to use spherical coordinates (r, θ, ϕ), where θ is the angle made by the radius vector with a fixed axis, and ϕ is the angle measured around this axis. By virtue of the spherical symmetry $\sigma_\theta = \sigma_\phi$ everywhere in the shell, due to spherical symmetry of the structure, the components of displacement in spherical coordinates (r, θ, ϕ) are given by $u = r(1-\beta)$, $v = 0$, $w = 0$, where u, v, w (displacement components); β is position function depending on $r = \sqrt{x^2+y^2+z^2}$ only.

The generalized components of strain are given by Seth /11, 12/ as

$$e_{rr} = \frac{1}{n}\left[1-(r\beta'+\beta)^n\right],$$
$$e_{\theta\theta} = e_{\phi\phi} = \frac{1}{n}\left[1-\beta^n\right], \tag{1}$$
$$e_{r\theta} = e_{\theta\phi} = e_{\phi r} = 0\,.$$

Stress-strain relation: the constitutive equation of stress-strain for isotropic material is given as /13/:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij}\,, \quad (i,j=1,2,3) \tag{2}$$

where: T_{ij} are the stress components; λ and μ are Lame's constants; $I_1 = e_{kk}$ is the first strain invariant; δ_{ij} is the Kronecker delta.

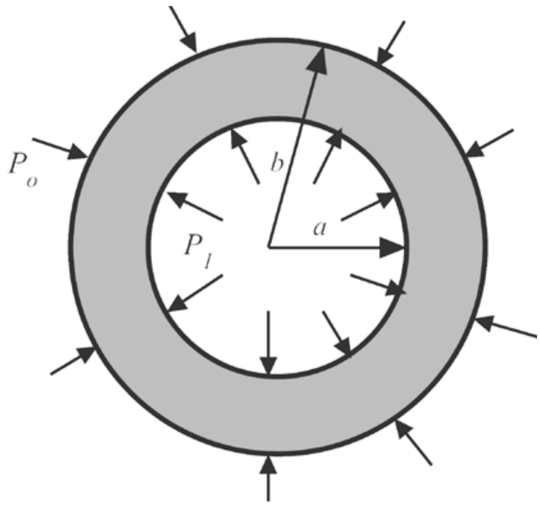

Figure 1. Geometry of spherical shell under internal and external pressure.

By using Eqs.(1) in Eq.(2), the stresses are obtained as:

$$T_{rr} = \frac{\lambda}{n}\left[3-2\beta^n-(r\beta'+\beta)^n\right]+\frac{2\mu}{n}\left[1-(r\beta'+\beta)^n\right],$$
$$T_{\theta\theta} = \frac{\lambda}{n}\left[3-2\beta^n-(r\beta'+\beta^n)\right]+\frac{2\mu}{n}\left[1-\beta^n\right]=T_{\phi\phi}\,, \tag{3}$$
$$T_{r\theta} = T_{\theta\phi} = T_{\phi r} = 0\,.$$

The radial equilibrium of an element of the spherical shell requires:

$$\frac{dT_{rr}}{dr} = \frac{2}{r}(T_{\theta\theta}-T_{rr})\,, \tag{4}$$

where: T_{rr} and $T_{\theta\theta}$ are the radial and hoop stresses. For sufficiently small values of the pressure, the deformation of the shell is purely elastic. The boundary conditions of the problem are:

$$T_{rr} = -p_1, \quad \text{at} \quad r = a$$
$$T_{rr} = -p_0, \quad \text{at} \quad r = b. \tag{5}$$

Using Eqs.(3) in Eq.(4), we get a nonlinear differential equation in β as:

$$P(P+1)^{n-1}\beta\frac{dP}{d\beta}+P(P+1)^n+2(1-c)P-\frac{2c}{n\beta^n}\times$$
$$\times\left[\left\{1-\beta^n(P+1)^n\right\}-(1-\beta^n)\right]=0, \tag{6}$$

where: compressibility $c = 2\mu/\lambda + 2\mu$ and $r\beta' = \beta P$ (P is function of β and β is function of r). The transition points of β in Eq.(6) are $P = 0$, $P \to -1$, and $P \to \pm\infty$. Hereby, we are only interested in finding plastic stresses corresponding to $P \to \pm\infty$.

DETERMINATION OF STRESSES IN ELASTIC-PLASTIC TRANSITION

In these types of problems, the solution of the problem depends upon the nature of the problem. The problem is based on the elastic-plastic transition, so we define a transition function through principal stresses T_{rr} or $T_{\theta\theta}$. Here, in order to calculate elastic-plastic stresses, we define the transition function by taking the principal stress T_{rr} (see Sharma /4-6/, Thakur /14-16/, Verma /17-20/) at the transition point $P \to \pm\infty$. The transition function Ψ is given as:

$$\Psi = (3\lambda + 2\mu) - nT_{rr} = \frac{2\mu\beta^n}{c}\left[(P+1)^n + 2(1-c)\right]. \quad (7)$$

Taking the logarithmic differentiating of Eq.(7) with respect to r and substituting the value of $dP/d\beta$ from Eq.(6) and taking the asymptotic value $P \to \pm\infty$, after integration we get:

$$\Psi = A_1 r^{-2c}, \quad (8)$$

where: A_1 is constant of integration.

By using Eqs.(7) and (8), we have the transition value $\boldsymbol{T_{rr}}$ as

$$T_{rr} = \frac{1}{n}(A_2 + A_1 r^{-2c}), \quad (9)$$

where: $A_2 = (3\lambda + 2\mu)$ is a constant.

By using the boundary conditions, we have

$$A_1 = \frac{nb^{2c}(p_1 - p_o)}{(b/a)^{2c} - 1}, \qquad A_2 = -np_2 + \frac{n(p_1 - p_o)}{(b/a)^{2c} - 1}. \quad (10)$$

Substuiting the values of the constants A_1, A_2 in Eq.(9), we have

$$T_{rr} = \frac{(p_1 - p_0)}{(b/a)^{2c} - 1}\left\{1 - \left(\frac{b}{r}\right)^{2c}\right\} - p_0. \quad (11)$$

By using the Eq.(11) in Eq.(4)

$$T_{\theta\theta} = T_{rr} + \frac{(p_1 - p_o)c}{(b/a)^{2c} - 1}\left(\frac{b}{r}\right)^{2c}. \quad (12)$$

Initial yielding: it is clear from Eq.(12) that the value of $|T_{\theta\theta} - T_{rr}|$ is maximal at $r = a$. Therefore, the yielding of the spherical shell take place at the internal surface

$$\left|T_{\theta\theta} - T_{rr}\right|_{r=a} = \left|\frac{(p_1 - p_o)c}{(b/a)^{2c} - 1}\left(\frac{b}{a}\right)^{2c}\right| \equiv Y. \quad (13)$$

Therefore, effective pressure required for initial yielding at the internal surface is given as

$$P_e = \frac{p_1}{Y} - \frac{p_0}{Y} = p_{i1} - p_{i0} = \frac{1 - (b/a)^{-2c}}{c}. \quad (14)$$

Fully-plastic state: for the fully plastic state, we make $c \to 0$ in Eqs.(14) and we get the following equations

$$\left|T_{\theta\theta} - T_{rr}\right|_{r=b} = \left|\frac{(p_1 - p_o)}{2\log(b/a)}\right| \equiv Y_1. \quad (15)$$

Therefore, the effective pressure and stresses required for fully plastic state at the external surface are given as

$$P_e = \frac{p_1}{Y_1} - \frac{p_0}{Y_1} = p_{f1} - p_{f0} = 2\log(b/a), \quad (16)$$

$$T_{rr} = (p_1 - p_0)\frac{\log\frac{r}{b}}{\log\frac{b}{a}} - p_0, \quad (17)$$

$$T_{\theta\theta} = T_{rr} + \frac{(p_1 - p_0)}{2\log(b/a)}. \quad (18)$$

RESULTS AND DISCUSSION

From the above investigation of introductory yielding and completely plastic state of the spherical shell, the effective pressure is calculated for both stages. Curves are plotted between effective pressure along the radii ratio b/a for the spherical shell made of compressible material as well as incompressible material. It is seen that the effective pressure value is maximum for low compressibility value $c = 0.15$ and this value further increases with increase in radii ratio b/a. It implies that the thick spherical shell yields at high pressure as compared to the thin spherical shell. As we increase the value of compressibility of the material, the value of effective pressure also decreases for thin as well as thick spherical shells.

It has been shown that the spherical shell having lower values of the compressibility requires high pressure to start initial yielding in the shell as compared to the spherical shell with high compressibility (see Fig. 2).

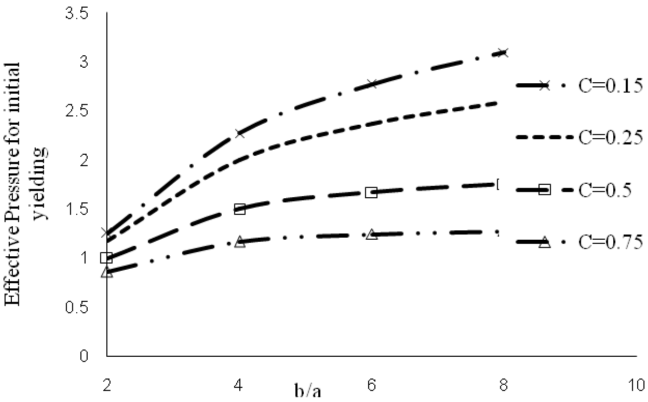

Figure 2. Effective pressure required for initial yielding in the spherical shell.

In Table 1, the stresses are calculated along the radii ratio r/b of the spherical shell under various values of internal and external pressure. In case 1, external pressure is taken greater than internal pressure. The negative value of radial and circumferential stresses shows that stresses are compressible in nature due to high external pressure. In case 2, external pressure is taken less than internal pressure. The positive value of radial and circumferential stresses shows that stresses are tensile in nature due to high internal pressure. In case 3, the internal pressure is taken as zero and stresses occur in the spherical shell are compressible due to the external pressure only, whereas in case 4, the external pressure is taken as zero and stresses occur in the spherical shell are tensile due to internal pressure only.

Table 1. Stress analysis of spherical shell under internal and external pressure.

$R=r/b$	p_1	p_0	p_1-p_0	Compress-ibility C	Radial stress	Circumfer-ential stress
Case 1: external pressure (p_0) > internal pressure (p_1)						
0.25	15	45	-30	0.25	-126.48	-127.23
0.5	15	45	-30	0.25	-135.533	-137.033
0.75	15	45	-30	0.25	-144.586	-146.836
1	15	45	-30	0.25	-153.64	-156.64
0.25	15	45	-30	0.5	-82.5	-85
0.5	15	45	-30	0.5	-90	-95
0.75	15	45	-30	0.5	-97.5	-105
1	15	45	-30	0.5	-105	-115
0.25	15	45	-30	0.75	-67.5604	-72.3818
0.5	15	45	-30	0.75	-73.7132	-83.3561
0.75	15	45	-30	0.75	-79.866	-94.3303
1	15	45	-30	0.75	-86.0189	-105.305
Case 2: internal pressure (p_1) > external pressure (p_0)						
$R=r/b$	p_1	p_0	p_1-p_0	Compress-ibility C	Radial stress	Circumfer-ential stress
0.25	45	15	30	0.25	66.47971	67.22971
0.5	45	15	30	0.25	75.53301	77.03301
0.75	45	15	30	0.25	84.58631	86.83631
1	45	15	30	0.25	93.63961	96.63961
0.25	45	15	30	0.5	22.5	25
0.5	45	15	30	0.5	30	35
0.75	45	15	30	0.5	37.5	45
1	45	15	30	0.5	45	55
0.25	45	15	30	0.75	7.560374	12.3818
0.5	45	15	30	0.75	13.7132	23.35606
0.75	45	15	30	0.75	19.86603	34.33032
1	45	15	30	0.75	26.01886	45.30458
Case 3: under external pressure only						
$R=r/b$	p_1	p_0	p_1-p_0	Compress-ibility C	Radial stress	Circumfer-ential stress
0.5	0	15	-15	0.25	-60.2665	-61.0165
0.75	0	15	-15	0.25	-64.7932	-65.9182
1	0	15	-15	0.25	-69.3198	-70.8198
0.25	0	15	-15	0.5	-33.75	-35
0.5	0	15	-15	0.5	-37.5	-40
0.75	0	15	-15	0.5	-41.25	-45
1	0	15	-15	0.5	-45	-50
0.25	0	15	-15	0.75	-26.2802	-28.6909
0.5	0	15	-15	0.75	-29.3566	-34.178
0.75	0	15	-15	0.75	-32.433	-39.6652
1	0	15	-15	0.75	-35.5094	-45.1523
Case 4: under internal pressure only						
$R=r/b$	p_1	p_0	p_1-p_0	Compress-ibility C	Radial stress	Circumfer-ential stress
0.5	15	0	15	0.25	45.2665	46.0165
0.75	15	0	15	0.25	49.79315	50.91815
1	15	0	15	0.25	54.31981	55.81981
0.25	15	0	15	0.5	18.75	20
0.5	15	0	15	0.5	22.5	25
0.75	15	0	15	0.5	26.25	30
1	15	0	15	0.5	30	35
0.25	15	0	15	0.75	11.28019	13.6909
0.5	15	0	15	0.75	14.3566	19.17803
0.75	15	0	15	0.75	17.43302	24.66516
1	15	0	15	0.75	20.50943	30.15229

In Fig. 3, shown is the distribution of elastic-plastic stresses in the spherical shell under impact of internal and external pressure. Stresses are plotted against the radii proportion r/b with the compressibility factor c = 0.25, 0.50, 0.75. It is observed from Fig. 3, that when $p_1 > p_0$, the stresses calculated are maximal at the exterior surface of the spherical shell and tensile in nature. When $p_0 > p_1$, the calculated stresses are maximal at the interior surface of the spherical shell and compressible in nature.

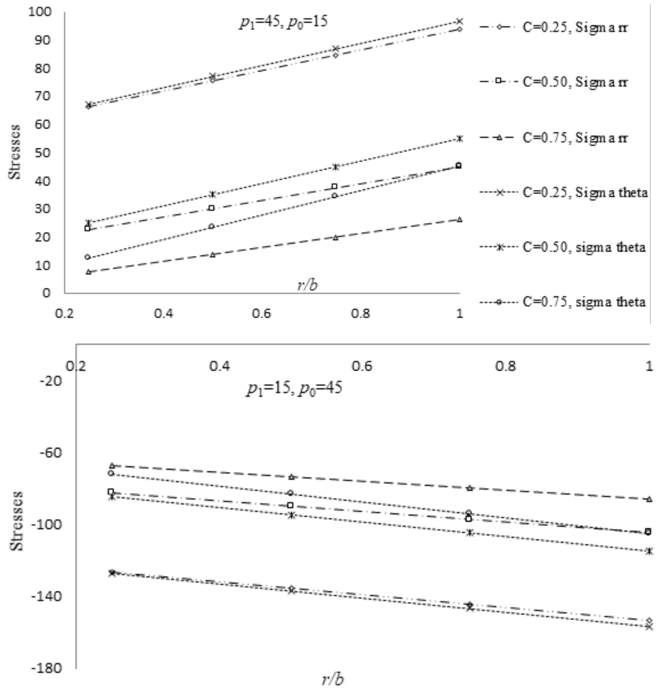

Figure 3. Dissemination of elastic-plastic stresses in the spherical shell under effects of internal and external pressure.

In Fig. 4, the independent effect of external pressure and internal pressure is shown on the spherical shell.

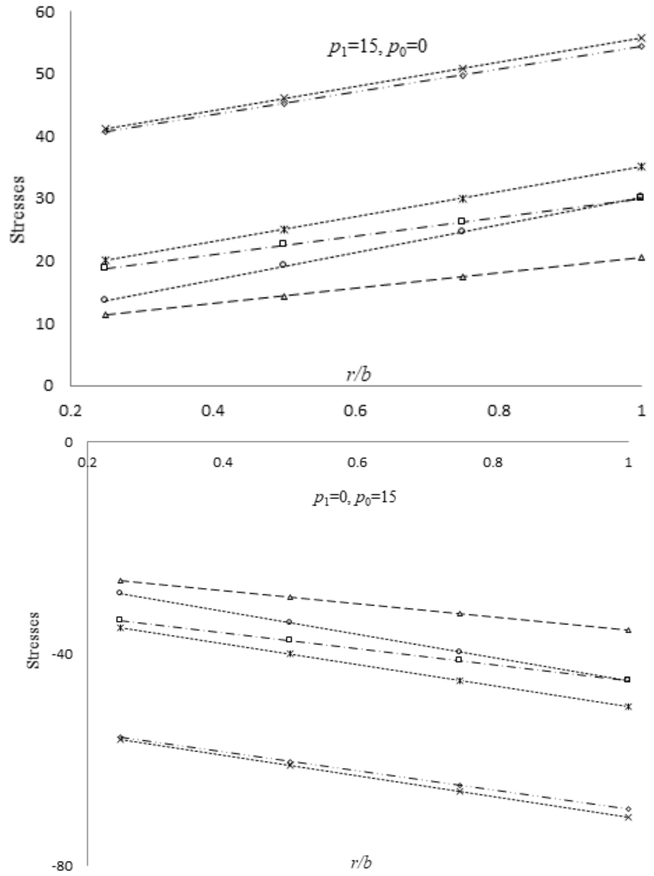

Figure 4. Dissemination of elastic-plastic stresses in the spherical shell under independent effects of internal and external pressure.

CONCLUSION

The solution of problem in the paper concludes that the incompressible material of the spherical shell entails high pressure for elasticity and is secure from the perspective of the design of spherical shell in contrast to the spherical shell of high compressibility under pressure. It happens because the proportionate increase in net pressure required for preliminary elasticity to turn out to be entirely plastic is highest for the incompressible material in comparison to compressible materials which leads to the durability and stability of the spherical shell and reduces the possibility of damage of spherical shell due to inner and outer pressure.

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