CREEP STRESSES AND STRAIN RATES FOR A TRANSVERSELY ISOTROPIC DISC HAVING THE VARIABLE THICKNESS UNDER INTERNAL PRESSURE

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Abstract

The paper presents the study of creep stresses and strain rates for a transversely isotropic disc having variable thickness, subjected to internal pressure, by using Seth’s transition theory. It has been observed that the disc made of variable thickness for transversely isotropic material has a maximum circumferential stress at the outer surface in comparison to disc having constant thickness and this value further increases with the increase in measure N and K. Strain rates are maximum on the internal surface for flat disc (k  = 0) but strain rates decrease at the internal surface for measure n  > 1. The strain rate for flat disc (k  = 0) further increases with increase in pressure at the internal surface. The disc of variable thickness (k  = 1.5) made of isotropic material has strain rate maximum at the external surface for P_i = 0.1 and measure n  = l. These values of strain rates further increase at the external surface with the increase in pressure and variable thickness ratio, but decrease with the increase in measure N.

INTRODUCTION

Several researchers have analysed the circular disc with constant material properties under various conditions. Solutions for thin isotropic discs can be found in most of the standard elasticity, plasticity and creep books. Disc plays an important role in machine design. Stress analysis of rotating discs has an important role in engineering design and structural engineering. Rotating discs are the most critical part of rotors, turbine motors, compressors, high speed gears, fly-wheel, sink fits, turbo jet engines and computer disc drives etc. Kirkner /6/ discussed the problem if vibration of a rigid disc on a transversely isotropic elastic half space. Mahalanabis et al. /7/ discussed the external thermal crack problem in a transversely isotropic elastic medium. Arnold /8/ has studied thermo-elastic transversely isotropic thick-walled cylinder/disk. Chen et al. /10/ discussed the problem of three-dimensional analytical solution for a rotating disc of functionally graded materials with transverse isotropy. Ghadi et al. /14/ discussed the problems of forced vertical vibration of rigid circular disc on a transversely isotropic half-space. Akinola et al. /16/ analysed the problem of large deformation of transversely isotropic elastic thin circular disc in rotation. Thakur /17/ has investigated the problem of infinitesimal deformation in a transversely isotropic thin rotating disc with rigid shaft by using Seth’s transition theory. Seth’s transition theory does not require these assumptions and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. Seth’s transition theory utilizes the
concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems /1, 2, 4, 5, 9, 11, 12, 13, 15, 17-30/.

GOVERNING EQUATIONS

We consider a thin disc having variable thickness and made of transversely isotropic material and having inner radius \( a \) and outer radius \( b \) respectively, subjected to internal pressure \( p \). The disc is thin and it is effectively in state of plane stress, i.e. axial stress \( T_{zz} \) is zero. The components of displacement are given by /1, 2/:

\[
u = r(1 - \beta_1), \quad v = 0, \quad w = 0 \tag{1}\]

where: \( u, v, w \) are displacement components; \( \beta \) is the position function, depending on \( r \). The generalized components of strain are given by /2/:

\[
\begin{align*}
\varepsilon_{rr} &= \frac{1}{n}[1 - (r \beta^p + \beta^p)], \\
\varepsilon_{zz} &= \frac{1}{n}[1 - (1 - d)^{2p}], \\
\varepsilon_{\theta\theta} &= \frac{1}{n}[1 - \beta^p], \\
\varepsilon_{\theta z} &= \frac{1}{n}[\beta^p - \beta^p], \\
\varepsilon_{\theta r} &= \frac{1}{n}[\beta^p - \beta^p].
\end{align*} \tag{2}\]

where: \( n \) is measure and \( \beta^p = d\beta/dr \).

**Stress-strain relation:** the stress-strain relations for transversely isotropic material are given:

\[
\begin{align*}
T_{rr} &= c_{11}c_{rr} + c_{12}c_{r\theta} + c_{13}c_{zz}, \\
T_{\theta\theta} &= c_{11}c_{\theta\theta} + c_{12}c_{\theta r} + c_{13}c_{\theta z}, \\
T_{\theta z} &= T_{z\theta} = T_{r\theta} = T_{zz} = 0.
\end{align*} \tag{3}\]

Substituting the strain components from Eq.(2) in Eq.(3), the stresses are obtained as:

\[
\begin{align*}
T_{rr} &= A_1 \left[ 2 - \beta^p \left( 1 + (1 + p)^p \right) \right] - 2c_{26}c_{zz} \left[ 1 - \beta^p \right] \\
T_{\theta\theta} &= A_1 \left[ 2 - \beta^p \left( 1 + (1 + p)^p \right) \right] - 2c_{26}c_{rr} \left[ 1 - \beta^p \left( 1 + p \right)^p \right] \\
T_{zz} &= T_{r\theta} = T_{\theta z} = T_{r\theta} = T_{zz} = 0
\end{align*} \tag{4}\]

where: \( A_1 = c_{11} - c_{12}^2 / c_{33} \) and \( r \beta^p = \beta^p P \).

**Equation of equilibrium:** the equation of equilibrium are all satisfy except

\[
\frac{d(hrT_{rr})}{dr} - hT_{\theta\theta} = 0 \tag{5}\]

where: \( T_{rr} \) and \( T_{\theta\theta} \) are the radial and hoop stresses, respectively.

**Critical points or turning points:** using Eq.(4) in Eq.(5), we get a nonlinear differential equation in \( \beta \) as:

\[
P \beta^{p+1} (1 + p)^{-1} \frac{dP}{d\beta} = \frac{2c_{26}c_{\theta\theta}}{n_1 h} \beta^p \left[ 1 - (1 + p)^p \right] - \frac{2c_{26}c_{zz}h'}{n_1 h} \left[ 1 - \beta^p \right] - P \beta^p \left[ 1 + (1 + p)^p \right] + \frac{2c_{26}c_{zz}h'}{A_1} \beta^p + \frac{2h' \beta^p}{n_1 h} \left[ 1 - \beta^p \right] + 1
\tag{6}\]

where: \( h' = dh/dr \) and \( r \beta^p = \beta^p P \) (\( P \) is the function of \( \beta \) and \( \beta \) is a function of \( r \) only). The transition points or turning point of \( \beta \) in Eq.(6) are \( P \to 1 \) and \( P \to \pm \infty \).

**Boundary conditions:** the boundary conditions are

\[
T_{rr} = -p \text{ at } r = a, \quad T_{rr} = 0 \text{ at } r = b. \tag{7}\]

**ANALYTICAL SOLUTION**

For finding the creep stresses and strain rates, the transition function is taken through principal stress difference (see /1, 2, 4, 5, 9, 11-13, 15, 17-30/) at the transition point \( P \to \pm \infty \). We define the transition function \( \zeta \) as:

\[
\zeta = T_{rr} - T_{\theta\theta} = \frac{2c_{26}c_{zz}h'}{n_1 h} \left[ 1 - (P + 1)^p \right] \tag{8}\]

where: \( \zeta \) is the function of \( r \) only.

Taking the logarithmic differentiation of Eq.(8) with respect to \( r \) and substituting the value of \( dP/d\beta \) from Eq.(6), we get:

\[
\frac{d}{dr} \left( \ln \zeta \right) = \frac{2c_{26}c_{zz}h'}{n_1 h} \left( 1 - 2n + C_2 (n - 1) \right) + \frac{2c_{26}c_{zz}h'}{n_1 h} \left( C_2 \right) - \frac{2c_{26}c_{zz}h'}{A_1} \beta^p + \frac{h' \beta^p}{n_1 h} \left( 1 - (P + 1)^p + 1 \right) \tag{9}\]

Taking asymptotic value of Eq.(9) as \( P \to \pm \infty \), we get

\[
\frac{d}{dr} \left( \ln \zeta \right) = \frac{1}{r} \left[ -2n + C_2 (n - 1) \right] + \frac{2c_{26}c_{zz}h'}{h} \frac{C_2 - 2}{h} \frac{h'}{h} \tag{10}\]

Asymptotic value of \( \beta \) as \( P \to 1 \) is \( D/r; D \) is a constant. Integrating Eq.(10), we get

\[
\zeta = T_{rr} - T_{\theta\theta} = A_2 \frac{e^h}{h} \exp f \tag{11}\]

where: \( A_3 = -2n + C_2 (n - 1) \), \( A_4 = 1 - C_2 \), \( C_2 = \frac{2c_{26}c_{zz}}{A_1} \), \( f = (2 - C_2) \frac{k^p}{h} \), and \( A_2 \) is constant of integration. Using Eq.(11) in Eq.(5), we get

\[
hT_{rr} = A_0 - A_5 \frac{hFdr}{} \tag{12}\]

where: \( A_5 = A_2 h_0^{1-\hat{A}_4} \), \( F = e^{-\frac{h}{h}} \left( \frac{r}{b} \right)^{-k(l-\hat{A}_4)} \) exp \( f \), and \( A_6 \) is another integration constant. Using Eq.(7) in Eq.(12), we get

\[
\frac{-ph_0}{a} \left( \frac{b}{a} \right)^k \quad \text{and} \quad \frac{-ph_0}{b} \left( \frac{b}{a} \right)^k \left[ \frac{Fdr}{a} \right] \quad \text{at} \quad r = b \tag{13}\]

Substituting Eq.(13) in Eq.(12), we get

\[
T_{rr} = \frac{ph(a)}{b} \left[ \frac{Fdr}{a} \right] \quad \text{at} \quad r = a \quad \text{and} \quad \frac{ph(a)}{b} \left[ \frac{Fdr}{a} \right] \quad \text{at} \quad r = b \tag{14}\]

Equation (14) gives creep stresses for a transversely isotropic disc with variable thickness under internal pressure. We introduce the following non-dimensional quantities: \( R = r/b \), \( R_0 = a/b \), \( c_{rr} = T_{rr}/p \), \( c_{26} = T_{\theta\theta}/p \), \( P_1 = c_{26}/c_{zz} \). Stresses Eq.(14) in non-dimensional form are given by:
\[
\sigma_{rr} = \frac{-R^{-k} R^k}{1} \int_{r_0}^{R} r^{-1-k(1+A_k)} \exp f_1 \, dr
\]
\[
\sigma_{\theta\theta} = \sigma_{rr} + \frac{-R^{-k} R^{k+1}}{1} \int_{r_0}^{R} r^{-1-k(1+A_k)} \exp f_1 \, dr
\]
where: \( f_1 = \left[ \frac{2}{c_{11} c_{33} c_{11}} \right] k \left( \frac{b R}{n D} \right) \), \( A_4 = \left( 1 - C_2 \right) \) and \( A_3 = -2n + C_2(n - 1) \).

**Flat disc:** for a disc having uniform thickness \((k = 0)\), Eq.(15) becomes:
\[
\sigma_{rr} = -\frac{1 - R^{-2n + C_2(n - 1)}}{1 - R_0^{-2n + C_2(n - 1)}}
\]
\[
\sigma_{\theta\theta} = \frac{1 - 2n + C_2(n - 1)}{1 - R_0^{-2n + C_2(n - 1)}} \left( R^{2n - 2} - 1 \right)
\]  
(16)

**Particular case:** Eq.(15) for isotropic material becomes
\[
\sigma_{rr} = \frac{-R^{-k} R^k}{1} \int_{r_0}^{R} r^{-1-k(1+A_k)} \exp f_2 \, dr
\]
\[
\sigma_{\theta\theta} = \frac{-R^{-k} R^{k+1}}{1} \int_{r_0}^{R} r^{-1-k(1+A_k)} \exp f_2 \, dr
\]  
(17)

where: \( f_2 = \left( \frac{3 - 2C}{2 - C} \right) k \left( \frac{b R}{n D} \right) \), \( A_7 = -2n + \frac{1}{2-C}(n-1) \), \( A_6 = \frac{2-C}{2} \), and \( C = \frac{2-\mu}{\lambda + 2\mu} \).

**Flat disc (isotropic case):** for a disc having uniform thickness \((k = 0)\), Eq.(17) becomes
\[
\sigma_{rr} = -\frac{1 - R^{-2n + n - 1}}{2-C}
\]
\[
\sigma_{\theta\theta} = \frac{1 - 2n + n - 1}{2-C} \left( R^{2n - 2} - 1 \right)
\]  
(18)

Equation (18) is similar as given by /12/.

**ESTIMATION OF CREEP PARAMETERS**

When the creep sets in, the stress-strain is given:
\[
\dot{\varepsilon}_{ij} = A_{ij} \frac{2C_{66}}{H} \delta_{ij} \Theta - 2 A_{ij} - C_{66} T_{ij}
\]  
(19)

where: \( \dot{\varepsilon}_{ij} \) is the strain component and \( T = T_r \) be first stress invariant and \( H = 4C_{66}C_{66} - A_{ij} \). When the creep sets in, the strain should be replaced by strain rates. The stress-strain relation Eq.(19) becomes,
\[
\dot{\varepsilon}_{ij} = A_{ij} \frac{2C_{66}}{H} \delta_{ij} \Theta - 2 A_{ij} - C_{66} T_{ij}
\]  
(20)

Differentiating Eq.(2) with respect to time \( t \), we get:
\[
\dot{\varepsilon}_{\theta\theta} = -\beta t^{-1} \beta
\]  
(21)

For SWAINGER measure (i.e. \( n = 1 \)), Eq.(21) becomes:
\[
\dot{\varepsilon}_{\theta\theta} = \beta
\]  
(22)

where: \( \dot{\varepsilon}_{\theta\theta} \) is the SWAINGER strain measure.

From Eq.(8) the transition value \( \beta \) is given at the transition point \( P \rightarrow -1 \) by:
\[
\beta = \left( n / 2C_{66} \right)^{1/n} \left[ T_r - T_{\theta\theta} \right]^{1/n}
\]  
(23)

Using Eqs.(21)-(23) in Eq.(19), we get
\[
\dot{\varepsilon}_{rr} = \frac{P \psi}{4} \frac{C_{66} + \frac{C_{13}}{c_{11}}}{\frac{C_{13}}{c_{33} c_{11}} - 1 - \frac{C_{13}}{c_{33} c_{11}}} \left[ 1 - \frac{c_{13}}{c_{33} c_{11} c_{11}} \right] \sigma_{\theta\theta} - \frac{1 - \frac{C_{13}}{c_{33} c_{11}}}{\frac{C_{13}}{c_{33} c_{11} c_{11}}} \sigma_{rr}
\]  
(24)

where: \( P = p/C_{66} \) and \( \psi = \left[ \frac{nP}{2} \left( \sigma_{rr} - \sigma_{\theta\theta} \right) \right]^{1/n} \).

For isotropic materials, strain rates Eq.(24) becomes
\[
\dot{\varepsilon}_{rr} = \frac{P \psi(2-C)}{2(3-2C)} \left[ \frac{1}{2-C} R^{2n - 2} - 1 \right] \sigma_{\theta\theta} - \frac{1}{2-C} \sigma_{rr}
\]  
(25)

where: \( \psi = \left[ \frac{nP}{2} \left( \sigma_{rr} - \sigma_{\theta\theta} \right) \right]^{1/n} \) and \( P = p/\mu \). These are the constitutive equations used by Odquist /3/ for finding the creep stresses, provided we put \( n = 1/N \) and \( N \) be the measure.

**NUMERICAL RESULTS AND DISCUSSION**

For calculating creep stresses and strain rates on the basis of above analysis, the following values of measure \( n \), \( D \) and pressure \( P_1 \) have been taken \( n = 1, 1/3, 1/7 \) (i.e. \( N = 1, 3 \) and 7) and \( P_1 = p/C_{66} \) = 0.1, 1.0 and \( D = 1 \). Elastic constants \( c_{ij} \) have been given in Table 1 for isotropic material (i.e. brass, \( c = 0.33 \)) and transversely isotropic material (i.e. cadmium).
Table 1. Elastic constants $c_{ij}$ (in units of $10^{10}$ N/m$^2$).

<table>
<thead>
<tr>
<th>Material Type</th>
<th>$c_{11}$</th>
<th>$c_{12}$</th>
<th>$c_{13}$</th>
<th>$c_{33}$</th>
<th>$c_{44}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic material (brass, $\sigma = 0.33$ or $C_1 = 0.50$)</td>
<td>3.0</td>
<td>1.0</td>
<td>1.0</td>
<td>3.0</td>
<td>0.999</td>
</tr>
<tr>
<td>Transversely isotropic material (cadmium $C_2 = 0.40$)</td>
<td>11.0</td>
<td>4.04</td>
<td>3.83</td>
<td>4.69</td>
<td>1.56</td>
</tr>
</tbody>
</table>

Curves are present in Fig. 1, between stresses along the radii ratio $R = r/b$ for $k = 0$ and $2.0$ respectively. It has been seen from Figs. 2, that the disc having a constant thickness ($k = 0$) subjected to internal pressure and made of a transversely isotropic material (cadmium) has maximum circumferential stress at the outer surface for measure $n = 1/3$ and this value further increases at the outer surface with the increase in the measure $n = 1/7$ in comparison to disc made of isotropic material (i.e. brass). The disc made of variable thickness for transversely isotropic material has maximum circumferential stress at the outer surface in comparison to flat disc ($k = 0$) for measure $n = 1/3$ and $k = 2$, and this value further increases with increase in measure $n$ and thickness $k$. Curves are presented Figs. 2 and 3, between creep strain rate along the radius ($R = r/b$) for $n = 1$ and $1/7$, and pressure $P_1 = 0.1$ and 1.0 respectively. It has been observed from figures that strain rate are maximum on the internal surface for flat disc ($k = 0$) made of a transversely isotropic material for measure $n = 1$ and pressure $P_1 = 0.1$ but strain rate decrease at the internal surface for measure $n > 1$. The strain rate for flat disc further increases with increase in pressure at the internal surface. The disc having variable thickness ($k = 1.5$) made of isotropic material have strain rates maximum at the external surface for $P_1 = 0.1$ and measure $n = 1$. These values of strain rates further increase at the external surface with the increase in pressure and variable thickness ratio, but decrease with the increase in measure $n$.

Figure 1. Creep stresses of a transversely isotropic disc with variable thickness under internal pressure, i.e. $k = 0$ (constant thickness, top), $k = 2$ (bottom).
Figure 2. Creep strain rates for a transversely isotropic disc with variable thickness under internal pressure ($P_i = 0.1$) for measure $n = 1$ (top) and $n = 1/7$ (bottom).
Thermal creep analysis of pressurized thick-walled cylindrical... 

REFERENCES


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