INTRODUCTION

Rotating discs have a wide range of applications in engineering, such as high-speed gears, turbine rotors, compressors, flywheel and computer disc drives. The analytical procedures presently available are restricted to problems with the simplest configurations. The use of rotating disc in machinery and structural applications has generated considerable interest in many problems in the domain of solid mechanics. They usually operate at relatively high angular speeds and temperatures. Therefore, the prediction of long-term steady-state creep deformation is very important for these applications. Classical theories of creep start with the assumptions of constitutive equations for creep and the classical theories of plasticity need an extra relation called the yield condition in addition to the flow rules. The description of deformations in a solid subjected to external forces is thus given by a different set of equations for elastic, plastic and creep deformations. Solutions for thin isotropic discs can be found in most standard creep textbooks /1-4/. In most engineering application, the disc has to operate under elevated temperature and is simultaneously subjected to high stresses caused by disc rotation at high speed. As a result of severe mechanical and thermal loading, the disc material undergoes creep deformation, thereby affecting performance of the system /5-7/. In recent years, the problem of creep in rotating discs of functionally graded materials, subjected to severe mechanical and thermal load, has attracted interest of many researchers. Seth, /8/, investigated the transition theory of elastic-plastic deformation, creep and relaxation. Gupta et al. /7, 9-11/, analysed creep transition in a thin rotating disc and cylinder having various conditions. Thakur /12/, analysed creep transition stresses in a thin rotating disc with shaft by finite deformation under steady state temperature. Thakur et al. /13/, investigated thermal creep stresses and strain rates in a thin rotating disc with variable density by finite deformation under steady state temperature. Thakur et al. /13/, analys...
density. Wahl, /14/, has investigated creep deformation in rotating discs assuming small deformation, incompressibility condition, Tresca yield criterion, its associated flow rule and a power strain law. The necessity of increasing use of ad-hoc semi-empirical laws in the classical theory of elastic-plastic and creep transition lies in the fact that the latter does not recognize the existence of the transition state between elastic and plastic ones and then creep. We have shown in this research paper that assumptions of yield conditions in such problems become unnecessary once we recognize that the transition from plastic state to creep as explained by Seth is an asymptotic process, and that the transition state is a separate state which cannot be replaced by a yield surface as has always been done in the current literature. This treatment in the classical theory to divide two extreme properties of a material by a sharp line which is physically impossible. It has been clear from our work that identification of the transition state is basically important. There are, at present, three ways to identify the transition state. The most general one among all is the vanishing of the Jacobian of transformation from elastic to plastic state and plastic to creep. An invariant relation among the strain (stress) invariants is obtained from this condition and it is found that most of the yield conditions present in current literature are obtainable from it as special cases. Also our results include the Bauschinger's effect while the classical yield conditions fail to account for it. The classical theory of elasticity, plasticity and creep makes use of linear strain measure. But we have shown that transition fields are sub-harmonic (super-harmonic) fields and that they are nonlinear and non-conservative in character and hence it is very important that a nonlinear strain measure such as the Almansi measure should be used in the constitutive equation. The recognition of transition state or mid-zone as a separate state necessitates showing the existence of the constitutive equation for that state. In this context, we have used Seth's transition theory to obtain the stresses and strains in the transition state and the same may be obtained for the plastic state when a certain parameter \( C = 1 - 2 v l_{1} - v \), where \( v \) is the Poisson's ratio of the material, is made to approach zero. From these solutions the constitutive equations for both transition and plastic states are obtained, the latter takes the form of the Levy-von-Mises equation. In nature, transitions do occur frequently and the existing classical theory fails to explain them successfully. Thus, the transition theory, as it stands, now can be fruitfully exploited to explain a variety of physical phenomena and hence has a very wide application in all applied sciences. Seth's transition theory does not acquire any assumptions like an yield condition, incompressibility condition and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. This theory utilizes the concept of generalised strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems /7-13, 15-18/. The density is assumed to vary along the radius in the form:

\[
\rho = \rho_{0}(r / m)^{-m}
\]  

where, \( \rho_{0} \) is density at \( r = b \), and \( m \) are density parameters, respectively.

**GOVERNING EQUATIONS**

Consider a thin rotating disc with central bore of internal radius \( a \) and external radius \( b \), respectively. The disc produced of variable density material is mounted on a rigid shaft and applied load \( P_{0} \) at the internal surface. The disc is rotating with angular speed \( \omega \) about a central axis perpendicular to its plane. The thickness of the disc is assumed to be constant and is taken sufficiently small, so that the disc is effectively in a state of plane stress, that is, the axial stress \( \sigma_{zz} \) is zero. The temperature at the central bore of the disc is \( \Theta_{0} \).

**Displacement coordinate:** the displacement components in cylindrical polar co-ordinate are given by Seth /8, 15/:

\[
u = r(1 - \beta); \quad v = 0 \quad \text{and} \quad w = dz (2)
\]

where \( u, v, w \) (displacement components); \( \beta \) is the position function, depending on \( r = \sqrt{x^{2} + y^{2}} \) only, and \( d \) is a constant. Generalized components of strain are given by Seth /15/:

\[
e_{rr} = \frac{1}{n} \left[ 1 - (r \beta' + \beta)^{n} \right] \quad e_{\theta\theta} = \frac{1}{n} \left[ 1 - \beta^{n} \right] \quad e_{zz} = \frac{1}{n} \left[ 1 - (1 - d)^{n} \right] \quad e_{r\theta} = e_{\theta z} = e_{z r} = 0 \quad (3)
\]

where \( \beta' = d \beta' dr \).

**Stress-strain relation:** the stress-strain relations for thermo-elastic isotropic material are given by Parkus, /19/:

\[
\sigma_{ij} = \lambda \delta_{ij} l_{1} + 2 \mu e_{ij} - \xi \Theta \delta_{ij}, \quad i, j = 1, 2, 3
\]  

where \( \sigma_{ij} \) are the stress components; \( \lambda \) and \( \mu \) are Lamé's constants; \( l_{1} = e_{ii} \) is the first strain invariant; \( \delta_{ij} \) is the Kronecker's delta; \( \xi = \alpha(3 \lambda + 2 \mu) / \beta \), \( \alpha \) being the coefficient of thermal expansion; and \( \Theta \) is temperature. Further, \( \Theta \) has to satisfy:

\[
\nabla^{2} \Theta = 0 \Rightarrow \frac{d^{2} \Theta}{dr^{2}} + \frac{1}{r} \frac{d \Theta}{dr} = 0 \quad \text{or} \quad \frac{d \Theta}{dr} = k_{1} \frac{1}{r}
\]

\[
\Theta = k_{1} \log(r + k_{2}) \quad (5)
\]

where \( k_{1} \) and \( k_{2} \) are constants of integration and can be determined from the boundary condition. Substituting Eq.(3) in Eq.(4), the stresses are obtained as:

\[
\sigma_{rr} = \frac{2 \mu}{n} \left[ 3 - 2C - \beta^{n} \right] \left[ 1 - C + (2 - C)(P + 1)^{\frac{1}{n}} + \frac{n \bar{C} \xi \Theta}{2 \mu \beta^{n}} \right]
\]

\[
\sigma_{\theta\theta} = \frac{2 \mu}{n} \left[ 3 - 2C - \beta^{n} \right] \left[ 2 - C + (1 - C)(P + 1)^{\frac{1}{n}} + \frac{n \bar{C} \xi \Theta}{2 \mu \beta^{n}} \right]
\]

\[
\sigma_{zz} = \sigma_{r\theta} = \sigma_{\theta z} = \sigma_{z r} = 0
\]  

where \( C = \frac{1}{2} \left[ 1 - \frac{v}{1 - v} \right] \) is the ratio of specific heat at constant pressure and constant volume and \( \bar{C} \) is the average specific heat.

\[
\rho_{0} \left( \frac{r}{m} \right)^{-m}
\]  

\[
\Theta = k_{1} \log(r + k_{2})
\]  

\[
\sigma_{rr} = \frac{2 \mu}{n} \left[ 3 - 2C - \beta^{n} \right] \left[ 1 - C + (2 - C)(P + 1)^{\frac{1}{n}} + \frac{n \bar{C} \xi \Theta}{2 \mu \beta^{n}} \right]
\]

\[
\sigma_{\theta\theta} = \frac{2 \mu}{n} \left[ 3 - 2C - \beta^{n} \right] \left[ 2 - C + (1 - C)(P + 1)^{\frac{1}{n}} + \frac{n \bar{C} \xi \Theta}{2 \mu \beta^{n}} \right]
\]

\[
\sigma_{zz} = \sigma_{r\theta} = \sigma_{\theta z} = \sigma_{z r} = 0
\]  

\[\Theta = k_{1} \log(r + k_{2})\]  

\[\sigma_{rr} = \frac{2 \mu}{n} \left[ 3 - 2C - \beta^{n} \right] \left[ 1 - C + (2 - C)(P + 1)^{\frac{1}{n}} + \frac{n \bar{C} \xi \Theta}{2 \mu \beta^{n}} \right]
\]

\[\sigma_{\theta\theta} = \frac{2 \mu}{n} \left[ 3 - 2C - \beta^{n} \right] \left[ 2 - C + (1 - C)(P + 1)^{\frac{1}{n}} + \frac{n \bar{C} \xi \Theta}{2 \mu \beta^{n}} \right]
\]

\[\sigma_{zz} = \sigma_{r\theta} = \sigma_{\theta z} = \sigma_{z r} = 0\]
where $C$ is the compressibility factor of the material in term of Lamé’s constant, and are given by $C = 2\mu/\lambda + 2\mu$.

**Equation of equilibrium:** the equations of motion are all satisfied except

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho \omega^2 r^2 = 0$$  \hspace{1cm} (7)

where $\sigma_r$ and $\sigma_\theta$ are the radial and hoop stresses, respectively.

**Boundary conditions:** the temperature satisfying Laplace Eq.(5) with boundary condition:

$$\Theta = \Theta_0$$

and $\sigma_r = 0$ at $r = a$

$$\Theta = 0$$

where $L_0$ is the load and $\Theta_0$ is constant, given by Parkus /19/.

Substituting $k_1 = \frac{\Theta}{\log(a/b)}$ and $k_2 = \log b$ from Eq.(5), we get:

$$\Theta = \Theta_0 \frac{\log(r/b)}{\log(a/b)}$$ \hspace{1cm} (9)

**Critical points or turning points:** using Eq.(6) and Eq.(9) in Eq.(7), we get a nonlinear differential equation in $\beta$ as:

$$(2-C)\beta^{n+1} P(P+1)\beta^{n-1} \frac{dP}{d\beta} = \frac{\rho \omega^2 r^2}{2\mu} - \frac{n C \zeta \Theta_0}{2\mu} + \beta^2 \left[1 - (P+1)^n - \frac{n P}{1-C + (2-C)(P+1)^n}\right]$$

$$+ \frac{\rho \omega^2 r^2}{2\mu} \left[1 - (P+1)^n - \frac{n P}{1-C + (2-C)(P+1)^n}\right]$$

where $\Theta_0 = \Theta_0 / \log(a/b)$ and $\beta_r = \beta_r P$ is the function of $\beta$ and $\beta$ is a function of $r$. Transition points of $\beta$ in Eq.(10) are $P \rightarrow -1$ and $P \rightarrow \pm \infty$.

**FORMULATION AND ANALYTICAL CREEP SOLUTION**

For finding the thermal creep stresses and strain rates, the transition function is taken through principal stress difference, see /7, 12, 15, 16/ at the transition point $P \rightarrow 1$. We define the transition function $\zeta$ as:

$$\zeta = \sigma_r - \sigma_\theta = 2\mu \beta^n \frac{1}{n} \left[1 - (P+1)^n\right]$$ \hspace{1cm} (11)

where $\zeta$ is a function of $r$ only and $\zeta$ is the dimension.

Taking the logarithmic differentiation of Eq.(11) with respect to $r$ and substituting Eq.(10), and taking asymptotic value $P \rightarrow -1$, we get:

$$\frac{d}{dr} \left(\log \zeta\right) = -\frac{1}{r(2-C)} \left[n(3-2C) + 1\right]$$

$$+ \frac{n \rho \omega^2 r^{2+n}}{2\mu D^n} - \frac{n C \zeta \Theta_0}{2\mu b^n (2-C)}$$

\hspace{1cm} (12)

Asymptotic value of $\beta$ as $P \rightarrow -1$ is $Dr/r; D$ being a constant. Substituting Eq.(1) into Eq.(12) and after integration with respect to $r$, we get

$$\zeta = \sigma_r - \sigma_\theta = Ar^k \exp \left( fr^{n-m+2} + gr^n\right)$$ \hspace{1cm} (13)

where $A, f$ and $g$ are constants of integration, which can be determined by boundary conditions and $\nu = (1-C)/(2-C)$ Poisson ratio, $C \zeta = \alpha E(1-\nu)$, $k = -(n+1) + \nu(n-1)$.

$$g = \frac{\alpha \Theta_0 (1+\nu)}{D^n}$$

$$f = -\frac{n \omega^2 \rho_0 (1-\nu)}{2\mu D^n b^{m-n}} - \frac{n \omega^2 \rho_0 (1-\nu^2)}{ED^n b^{m-n}}$$

From Eqs.(11) and (13), we have

$$\sigma_r - \sigma_\theta = Ar^k \exp \left( fr^{n-m+2} + gr^n\right)$$ \hspace{1cm} (14)

Substituting Eq.(14) in Eq.(7), we get:

$$\sigma_r = B - A r^{k-1} \exp \left( fr^{n-m+2} + gr^n\right) dr - \frac{\rho_0 \omega^2 r^{2-m}}{(2-m)b^{m}}$$ \hspace{1cm} (15)

where $B$ is a constant of integration, which can be determined by boundary condition. Using boundary condition, Eq.(8) in Eq.(15), we get

$$A = \frac{1}{\int_a^b r^{-1} \exp \left( fr^{n-m+2} + gr^n\right) dr} \times$$

$$B = \frac{a^{b-1} \exp \left( fr^{n-m+2} + gr^n\right) dr}{\int_a^b r^{-1} \exp \left( fr^{n-m+2} + gr^n\right) dr}$$

Substituting the values of constants of integration, $A$ and $B$ in Eq.(15), we get:

$$\sigma_{\theta} = \left[ L_0 - \frac{\rho_0 \omega^2 (b^{2-m}-\alpha^2-m)}{(2-m)b^{m}} \right] \times$$

$$+ \frac{\rho_0 \omega^2 (b^{2-m}-\alpha^2-m)}{(2-m)b^{m}}$$

\hspace{1cm} (16)

Substituting Eq.(16) in Eq.(14), we get:

$$\sigma_{\theta} = \left[ L_0 - \frac{\rho_0 \omega^2 (b^{2-m}-\alpha^2-m)}{(2-m)b^{m}} \right] \times$$

$$+ \frac{\rho_0 \omega^2 (b^{2-m}-\alpha^2-m)}{(2-m)b^{m}}$$

\hspace{1cm} (17)

Equations (16)-(17) give creep stresses for a thin rotating disc with shaft at temperature $\Theta_0$. We introduce the following non-dimensional components as:

$$R = r/b, R_0 = a/b, \sigma_r = \sigma_r/R_0, \sigma_\theta = \sigma_\theta/R_0, \Omega = \omega \rho_0 b^{2-m}, \sigma_0 = \Omega, \zeta = \zeta/R_0, \Theta = \Theta_0/R_0$$

Equations (16)-(17) in non-dimensional form become:

$$\sigma_\theta = \left[ \sigma_0 - \frac{\Omega^2 (1-R^2-b^{2-m})}{2-m} \right] \times$$

\hspace{1cm} (18)

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Creep stresses in a rotating disc having variable density and …

\[
\begin{align*}
&\frac{1}{R} \int_{R}^{R_{0}} \exp(f_{1}R^{n-m+2} + g_{1}R^{n})dR - R^{2} \exp(f_{1}R^{n-m+2} + g_{1}R^{n})dR \\
&+ \frac{1}{k_{0}} \int_{R}^{R_{0}} \exp(f_{1}R^{n-m+2} + g_{1}R^{n})dR \\
&\Omega^{2}(1-R^{2-m}) \\
&\sigma_{t} = \left[ \sigma_{0} - \frac{\Omega^{2}(1-R_{0}^{2-m})}{2-m} \right] \times \\
&\frac{1}{R} \int_{R}^{R_{0}} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR - R^{2} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR \\
&+ \frac{1}{k_{0}} \int_{R}^{R_{0}} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR \\
&\Omega^{2}(1-R^{2-m}) \\
&\frac{1}{2-m}
\end{align*}
\]

where \( f_{1} = -n \Omega^{2}(1-v^{2})b^{n} \), \( g_{1} = \frac{\Theta(1+\nu)b^{n}}{D^{n} \ln R_{0}} \) (constants);
\( \sigma_{0} \) is tangential stress; \( \sigma_{t} \) is radial stress; \( R = r/b, R_{0} = a/b \) (radii ratios).

**Fully-plastic state:** for a disc made of incompressible material (\( \nu \rightarrow 1/2 \) or \( C = 0 \)) Eqs.(23) to (25) become:

\[
\begin{align*}
&\sigma_{0} = \left[ \sigma_{0} - \frac{\Omega^{2}(1-R_{0}^{2-m})}{2-m} \right] \times \\
&\frac{1}{R} \int_{R}^{R_{0}} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR - R^{2} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR \\
&+ \frac{1}{k_{0}} \int_{R}^{R_{0}} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR \\
&\Omega^{2}(1-R^{2-m}) \\
&\frac{1}{2-m}
\end{align*}
\]

\[
\begin{align*}
&\sigma_{t} = \left[ \sigma_{0} - \frac{\Omega^{2}(1-R_{0}^{2-m})}{2-m} \right] \times \\
&\frac{1}{R} \int_{R}^{R_{0}} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR - R^{2} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR \\
&+ \frac{1}{k_{0}} \int_{R}^{R_{0}} \exp(f_{2}R^{n-m+2} + g_{2}R^{n})dR \\
&\Omega^{2}(1-R^{2-m}) \\
&\frac{1}{2-m}
\end{align*}
\]

where \( f_{2} = -3n \Omega^{2}b^{n} \); \( k_{1} = \frac{3n+1}{2} \); and \( g_{2} = \frac{3 \Theta b^{n}}{2D^{n} \ln R_{0}} \).

**RESULTS AND DISCUSSION**

For calculating creep stresses, based on the above analysis, the following values have been taken \( \Omega^{2} = \rho \omega^{2}b^{2}/E = 50, 75, \nu = 0.5 \) (incompressible material i.e. rubber), \( \nu = 0.42857 \) (compressible material, i.e. saturated clay) and 0.333 (compressible materials i.e. copper), \( n = 1/3, 1/5, 1/7 \) (i.e. \( N = 3, 5, 7 \)), \( \alpha = 5.0 \times 10^{-5} \) deg F^{-1} (for methyl methacrylate, Levitsky, 1975), \( \sigma_{0} = 100, 200 \) and \( \Theta_{0} = 0.00 \) and 1.5 and \( D = 1 \).

In classical theory measure \( N \) is equal to 1/\( n \). Definite integrals in Eqs.(18)-(19) have been solved by using Simpson’s rule. Curves are produced between creep stresses along the radii ratio \( R = r/b \) (see Figs. 1a-h) for rotating disc made of compressible, as well as incompressible materials, with variable density \( m = 0.5, 1 \), and load 100, 200 at different angular speed \( \Omega^{2} = 50 \) and 75. It is also observed from Figs. 1a-h that the circumferential stress has maximum value at the internal surface for measure \( n = 1/3 \) for compressible materials whereas reverse results for measure \( n = 1/7 \) for the incompressible material. With the introduction of thermal effects and load, circumferential stress as well as the radial stress both increase at the inner and outer surface of the rotating disc. The rotating disc is likely to fracture by cleavage close to the shaft at the bore.

**REFERENCES**

Creep stresses in a rotating disc having variable density and load.

Figure 1a. Creep stresses in a thin rotating disc having density $m = 0.5$ and load $\sigma_0 = 100$ for compressible and incompressible materials at angular speed $\Omega^2 = 50$ along the radii ratio $R = r/b$.

Figure 1b. Creep stresses in a thin rotating disc having density $m = 0.5$ and load $\sigma_0 = 200$ for compressible and incompressible materials at angular speed $\Omega^2 = 50$ along the radii ratio $R = r/b$.

Creep stresses in a rotating disc having variable density and …

Figure 1c. Creep stresses in a thin rotating disc having density \( m = 0.5 \) and load \( \sigma_0 = 100 \) for compressible and incompressible materials at angular speed \( \Omega = 75 \) along the radii ratio \( R = r/b \).

Figure 1d. Creep stresses in a thin rotating disc having density \( m = 0.5 \) and load \( \sigma_0 = 200 \) for compressible and incompressible materials at angular speed \( \Omega = 75 \) along the radii ratio \( R = r/b \).
Creep stresses in a rotating disc having variable density and …

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Figure 1e. Creep stresses in a thin rotating disc having density \( m = 1 \) and load \( \sigma_0 = 100 \) for compressible and incompressible materials at angular speed \( \Omega^2 = 50 \) along the radii ratio \( R = r/b \).

Figure 1f. Creep stresses in a thin rotating disc having density \( m = 1 \) and load \( \sigma_0 = 200 \) for compressible and incompressible materials at angular speed \( \Omega^2 = 50 \) along the radii ratio \( R = r/b \).
Creep stresses in a rotating disc having variable density and load 

\[ \tau = 0.428333 \text{ (Compressible material)} \]

\[ \nu = 0.333 \text{ (Compressible material)} \]

\[ n = 1/3, \sigma_{\theta, \text{Temp}} = 0 \]
\[ n = 1/3, \sigma_{r, \text{Temp}} = 0 \]
\[ n = 1/7, \Sigma_{\theta, \text{Temp}} = 0 \]
\[ n = 1/7, \Sigma_{r, \text{Temp}} = 0 \]
\[ n = 1/5, \Sigma_{\theta, \text{Temp}} = 0 \]
\[ n = 1/5, \Sigma_{r, \text{Temp}} = 0 \]

\[ \nu = 0.5 \text{ (Incompressible material)} \]

\[ n = 1/3, \sigma_{\theta, \text{Temp}} = 0 \]
\[ n = 1/3, \sigma_{r, \text{Temp}} = 0 \]
\[ n = 1/7, \Sigma_{\theta, \text{Temp}} = 0 \]
\[ n = 1/7, \Sigma_{r, \text{Temp}} = 0 \]
\[ n = 1/5, \Sigma_{\theta, \text{Temp}} = 0 \]
\[ n = 1/5, \Sigma_{r, \text{Temp}} = 0 \]

\[ \Omega = 75 \text{ along the radii ratio } R = r/b. \]

Figure 1g. Creep stresses in a thin rotating disc having density \( m = 1 \) and load \( \sigma_0 = 200 \) for compressible and incompressible materials at angular speed \( \Omega = 75 \) along the radii ratio \( R = r/b. \)

Figure 1h. Creep stresses in a thin rotating disc having density \( m = 1 \) and load \( \sigma_0 = 100 \) for compressible and incompressible materials at angular speed \( \Omega = 75 \) along the radii ratio \( R = r/b. \)

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