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EFFECTS OF NOISE AND INTERNAL DELAY ON COHERENT OSCILLATIONS IN TWO STOCHASTICALLY PERTURBED DELAYED DYNAMICAL SYSTEMS

EFEKTI ŠUMA I UNUTRAŠNJEG KAŠNJENJA NA KOHERENTNE OSCILACIJE U SLUČAJU DVA STOHASTIČKA DINAMIČKA SISTEMA SA KAŠNJENJEM

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Keywords
- coherent oscillations
- noise
- internal delay
- coupling delay
- bifurcation

Abstract

Variations in coherence of spike trains produced by two stochastically perturbed delayed dynamical systems are numerically examined. Type II excitable systems with noise introduced in the first equation of our dynamical system are used, with a special coherence mechanism called self-induced stochastic resonance, or SISR. Interplay between internal delay, noise and coupling delay can increase coherence, and can also lead to quite incoherent spiking trains. Numerically observed results could be qualitatively explained by considering bifurcations in the system caused by variations of noise and both delays.

Ključne reči
- koherentne oscilacije
- šum
- unutrašnje kašnjenje
- kašnjenje u vezi
- račvanje

Izvod

U ovom radu je numerički ispitana varijacija u koherenciji nizova pikova koji su rezultat dva stohastički perturbovana dinamička sistema sa kašnjenjem. Koristili smo ekscitabilne sisteme tipa II sa šumom u njihovoj prvoj jednačini, sa specijalnim mehanizmom koherencije koji se naziva samo indukujuća stohastična rezonanca (SISR). Interaktivni odnos između unutrašnjeg kašnjenja, šuma i kašnjenja u vezi, može povećati koherenciju ali isto tako može dovesti i do vrlo inkoherentnih nizova pikova. Numerički dobijeni rezultati se kvalitativno objašnjavaju uzimajući u obzir bifurkacije u sistemu prouzrokovane varijacijom šuma, i oba kašnjenja.

INTRODUCTION

Understanding the effect of random perturbations on nonlinear dynamical systems is a challenge across many disciplines of science. These perturbations may be small and irrelevant, or may be so large as to overwhelm the dynamics. More interestingly, they can be small and yet result in profound qualitative changes in the system behaviour without introducing any significant randomness, /1/.

An important class of nonlinear dynamical systems in which this phenomenon may occur are excitable systems. Excitable systems arise in a wide variety of areas which include climate dynamics, semiconductors, chemical reactions, lasers, combustion, neural systems, cardiovascular tissues, etc. and are especially common in biology /2-4/. A canonical example of a biological excitable system is a nerve cell. The defining property of all these systems is the way they respond to perturbations. If a perturbation is sufficiently small, the system quickly relaxes back into the unique stable steady state. On the other hand, once the perturbation reaches a certain threshold, a large transient response, e.g. an action potential in nerve cells, is triggered before the system recovers to its steady state.

Noise-driven excitable systems can produce dynamical responses which possess a high degree of coherence and yet are significantly different from what is observed in the absence of noise. One mechanism by which this phenomenon can occur is coherence resonance CR. It was first proposed in the work of Pikovsky and Kurths /5/ and since then attracted considerable attention. In CR, a dynamical system near but before Hopf bifurcation threshold is driven by small noise towards the deterministic limit cycle which emerges right after the bifurcation. Recently, an alternative mechanism, termed self-induced stochastic resonance SISR, has been proposed, /6/. In SISR small random perturbations also lead to the emergence of a coherent limit cycle behaviour, but in a profoundly different way and with different properties than in CR.

Two kinds of model perturbations are quite common. The first one replaces the complicated and often unknown influences of the system's environment by some type of noise. The other source of the model perturbations is motivated by quite different time scales that characterize excitable systems. Internal dynamics of a single excitable unit, for example a single neuron, occurs on a much faster scale than the transport of excitations between the units. This justifies introduction of an explicit time delay in the terms describing the interaction. However, there is yet another quite fundamental difference in the time-scales characteristic of a single excitable unit. This is the difference between the time-scales of the dynamics of so-called excitatory (fast) and refractory (slow) variables. The first variable changes quite rapidly and is usually identified with a single, well defined and measurable physical characteristic of the system, like for example the membrane potential in the neuronal models. The dynamics of the second, refractory, variable usually only qualitatively corresponds to a collection of unspecified processes with quite slow dynamics, and its role is to complete the model of the excitable behaviour. It is plausible that an internal time-delay in the coupling between the excitatory and refractory variable is justified, and furthermore, one should expect that such time-delay will have important qualitative influence.

MODEL

Common types of excitable behaviour are of two different varieties, usually called type I and type II, /7/. Type II excitability is characterized by the Hopf Bifurcation of the equilibrium into stable oscillatory dynamics. Most common elementary model of this type of excitability is provided by the FitzHugh-Nagumo dynamical equations, /7/:

$$\begin{aligned} \varepsilon dx &= f(x,y)dt = (x - x^3/3 - y) \\ dy &= g(x,y)dt = (x+a)dt \end{aligned} \qquad (1)$$

where x and y are the fast excitatory and slow refractory variables respectively, and ε is a small parameter, here set to $\varepsilon = 10^{-2}$ which guaranties the time scale difference between $x(t)$ and $y(t)$. Physical interpretation of the x and y variable is of no importance, as the equations are never used to model a particular system, but the phenomenon of (type II) excitability which occurs in many different systems. In neuronal models x is usually related to the cell membrane potential of a single neuron, but could also represent a collective variable of a network of neurons exhibiting the excitable dynamics. The parameter a is the bifurcation parameter. For $|a| > 1$, the system, Eq.(1), is excitable and for stationary state is unstable and there exists a stable limit cycle. In this paper $a = 1.05$. The random process can be modelled by the additive white noise terms in the first and second equation of the system (1) in our case it will be in the first. The oscillations in our case, $D_1 \neq 0$ are induced by a mechanism that has been studied in detail, e.g. /6/, where it has been called self-induced stochastic resonance (SISR). The main properties of SISR (and the name) follow the fact that the system (1) asymptotically resembles a particle in a double well potential, /6/, and thus the coherent oscillations resemble the well-known effect of the stochastic resonance, /8, 9/. In particular, SISR happens even when a is far from the bifurcation value, and the resulting stochastic limit cycle does not resemble anything that could occur in the deterministic system. A pair of identical type II excitable noisy systems with the internal delays and delayed interaction is described by the following stochastic delay differential equations:

$$\begin{aligned} \varepsilon dx &= f\left(x_i(t), y_i(t-\tau_{in})\right)dt + \\ &\quad + c\left(x_i(t) - x_j(t-\tau)\right) + \sqrt{\varepsilon}\sqrt{2D_1^i}\,dW_1^i, \\ dy &= g\left(x_i(t) - y_i(t)\right)dt \end{aligned} \qquad (2)$$

Values of the excitability parameter a, and the internal delays τ_{in} in the two units will be always equal, and denoted by the same symbols. The refractory period of a single isolated unit for $a = 1.05$ is $T_{ref} \approx 1.4$. Values of time lags will be in the range of: $0 < \tau_{in} \leq 0.2$ for internal time lag, and $0 < \tau \leq 1.3$ for the interaction time lag, and will always be smaller than T_{ref} (small or large time-lag will mean relative to the refractory period). The noise intensity will be denoted as $D_1^{1,2}$ where the subscript corresponds to the equation of the first variable of either units. The superscript denotes the first or the second unit. If there is no superscript, the symbol is to apply to the both units.

There are almost no analytical methods to analyse statistical properties of a system of nonlinear stochastic delay differential equations. Due to the time-delay, the generated stochastic process is not Markovian so that the well-known techniques, like the equivalent Fokker-Plank equation, are not applicable. Nevertheless, there are some analytical methods, like generalization of Lyapunov functional on the problem of stochastic stability /10, 11/. Approximations by stochastic differential equations without time-delay, by treating the time-delay as small perturbation, lead to the replacement of the non-Markovian evolution with the Markovian, and in some cases analytical, treatment of the resulting Fokker-Plank equation can provide useful information, /12/. However, even the stationary solution of the Fokker-Plank equation for a single stochastic FitzHugh-Nagumo neuron is known only for special values of the parameters. There is no applicable analytic treatment of the second order statistic of coupled stochastic FitzHugh-Nagumo systems (with or without delay) which is needed for the description of noise induced coherence, /13/. Because of the lack of analytical methods, in order to study the statistical property of systems like Eq.(2), one is forced to rely on numerical computation.

NUMERICAL RESULTS

In this section we present results of extensive numerical computations performed in order to understand the interplay of internal delays and noise on the coherence properties of the excitable unit outputs. Independently of the main mechanism behind the occurrence of more or less regular sequence of spikes, the coherence of such spike trains can be quantified by a kind of signal-to-noise ratio defined by

$$SNR_{1,2} = \overline{T_k} / \sqrt{Var(T_k)} \qquad (3)$$

where: $T_k = t_k - t_{j-1}$ is the k-th inter-spike time interval and the overline, like in T_k, denotes time averaging. Large SNR corresponds to high coherence of the noise induced spike trains. Subscript will denote the first or the second unit. In our numerical integration we have used the Runge-Kuta 4-th order routine for the deterministic part of Eq.(2) and the Euler method for the stochastic part. Many sample paths for each value of the variable parameters $D_1^{1,2}$ have been calculated.

Results are compared with examples of other published work, /14/, and with performed computations using ready-made programs for solving SDDE's available within the XPP package, /15/.

We study the dependence of the coherence $\mathrm{SNR}_{1,2}$ on the internal time-lag and noise intensity, for fixed values of the coupling strength c and interaction time-delay τ. We shall then demonstrate that main qualitative properties of this dependence start to depend crucially on the values τ, only for sufficiently large $\tau > 1.15$. We concentrate on two excitable units with noise in the equations for the excitatory variable only. Furthermore, we fix the noise intensity of the first unit to $D_1^1 = 10^{-2}$, which roughly corresponds to the coherence maximum of SISR in the case of a single isolated unit with zero internal delay. The noise intensity in the second unit is variable and goes over the range $D_1^2 \in (10^{-4} - 10^{-1})$. As pointed out $c = 0.1$ and $\tau = 0.7$ are fixed and we compute $\mathrm{SNR}_{1,2}(D_1^2)$ for the first and second unit and for several fixed values of the internal delay, all quite small with respect to the refractory period, e.g. $\tau_{in} = 0$; 0.05; 0.1; 0.15; 0.2. A sample of the results of such calculations are shown in Fig. 1a, b.

On the basis of this and similar computations we can make following observations:

1) Internal time-delay τ_{in} can increase the coherence $\mathrm{SNR}_{1,2}$ like in Fig. 1.

2) The coherence $\mathrm{SNR}_{1,2}$ can be increased roughly uniformly in D_1^2 or can be strongly increased for smaller D_1^2 with much smaller effect for larger D_1^2 like in SNR_2. At the same time, the increase of SNR_1 for such τ_{in} is less.

3) Internal delay induces a significant change of the second unit coherence SNR_2, while the coherence $\mathrm{SNR}_2(D_1^2)$ decreases as D_1^2 is increased.

The large change of the coherence appears as an effect which is only deteriorated by noise. The computations for many fixed values of the internal delay, illustrated in Fig. 1, for only most characteristic values, are collected in Fig. 2. At each fixed value of the τ_{in} curves $\mathrm{SNR}_{1,2}(D_1^2)$ display maximal values at corresponding values of noise intensity D_1^2 and fixed $D_1^1 = 10^{-2}$. These maxima for many different τ_{in} are denoted $\mathrm{SNR}_{1,2}(\tau_{in})$ and displayed in Fig. 2. Quite sudden increase of the coherence $\mathrm{SNR}_2(\tau_{in})$ for the value of τ_{in}, roughly $\tau_{in} \approx 015$ is obvious.

We shall now provide a qualitative explanation for the variations of the coherence $\mathrm{SNR}_{1,2}(\tau_{in})$ illustrated in Figs. 1 and 2. The explanation is based on qualitative analyses of the dependence of the dynamics of the single unit (see for example /16, 17/) and of the pair on τ. When there is no internal delay $\tau_{in} = 0$ single excitable unit (1) with $a = 1.05$ has only one attractor in the form of the stable stationary solution, up to the critical interaction delay $\tau_c \approx 1.15$. A global fold limit cycle bifurcation occurs due to variation of the interaction delay at $\tau = \tau_c$. Up to the interaction delays of the order of the refractory period $\tau \leq 1.4$ there are two attractors: a stable stationary solution, and a stable limit cycle.

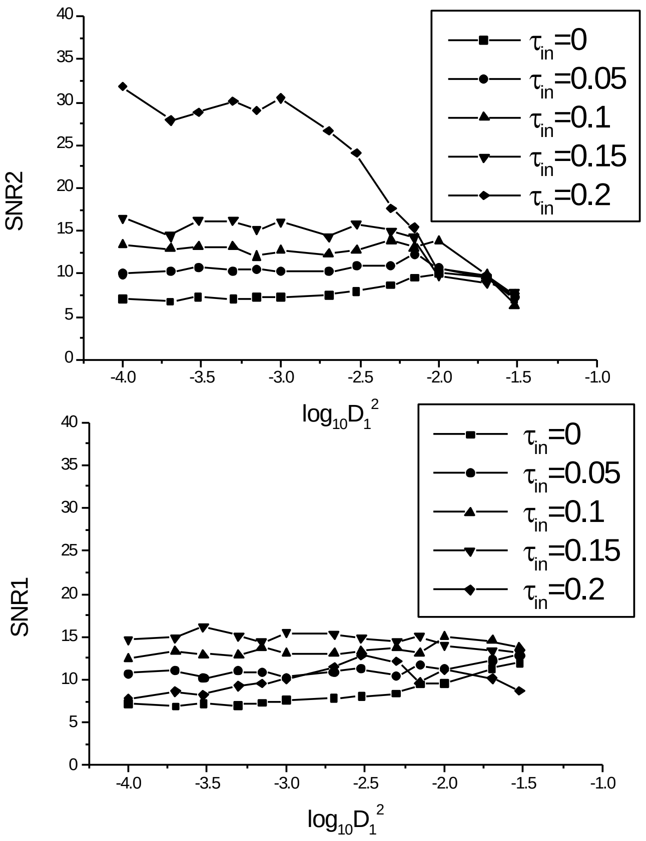

Figure 1. Illustrates dependence of the coherence: a) $\mathrm{SNR}_1(D_1^2)$, and b) $\mathrm{SNR}_2(D_1^2)$, on the variable noise intensity D_1^2 in the first equation of the second unit, for several fixed values of the internal delay τ_{in}. The fixed parameters are $c = 0.1$, $\tau = 0.7$, $D_1^1 = 10^{-2}$.

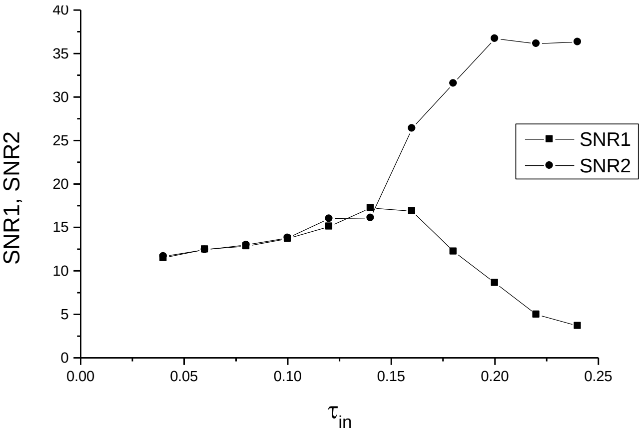

Figure 2. Illustrates $\mathrm{SNR}_{1,2}(D_1^2)$ for fixed $D_1^1 = 10^{-2}$ and representative values of D_1^2 as explained in the main text.

Unstable limit cycle also exists and acts as the threshold between the excitable and oscillatory asymptotic dynamics. The same behaviour occurs for sufficiently small, but nonzero internal delay. However, as the internal delay is increased at some values $\tau_{in} \approx 0.1$ global fold limit cycle bifurcation creates two limit cycles, a stable and an unstable one. As τ_{in} is increased the unstable limit cycle collapses onto the stable stationary state. The unstable cycle disappears and the stationary solution becomes unstable. This, subcritical Hopf bifurcation occurs for $\tau_{in} = \tau_{in,c} \approx 0.15$. For τ_{in} larger than this bifurcation value, the only attractor is the large stable limit cycle. The period of this periodic dynamics depends on the values of τ_{in} and increases with τ_{in}. The form of the limit cycle is of course that of the relaxation oscillator. Thus, for $\tau < \tau_c \approx 0.15$ and $\tau_{in} > \tau_{in,c} \approx 0.15$ there is one stable limit cycle, and for $\tau > \tau_c \approx 0.15$ and $\tau_{in} > \tau_{in,c} \approx 0.15$ there are two stable limit cycles.

Addition of noise has the following effect. For $\tau_{in} < \tau_{in,c} \approx 0.15$ and $\tau < \tau_c \approx 0.15$, and because of the proximity to the bifurcation point, addition of noise results in more or less regular series of spikes. Stochastic coherence can occur and is indicated by relatively large values of $SNR_{1,2}$. Immediately after the bifurcation point $\tau_{in} > \tau_{in,c}$, coherent oscillations on the large limit cycle dominate and lead to the sudden large increase of coherence $SNE_{1,2}$ irrespective of the noise intensity. Frequency of the oscillations is large and the time spent near the refractory states is short so that the noisy perturbation has small probability of inducing any significant number of spikes. The only effect of noise are small stochastic perturbations of the oscillation period. However, as the internal time delay is increased further, the period of the limit cycle oscillations and the time spent near the refractory period of the limit cycle oscillations, and the time spent near the refractory states, increases. Therefore, the probability of stochastically induced spikes in between two regular spikes increases, and the coherence of the resulting spike train decreases. This mechanism is relatively independent on the interaction as long as the interactions strength c and the interaction delays τ are such that no bifurcations occur due to c or τ.

The bifurcation sequence in the single unit caused by the internal delay, and the bifurcation caused by the external delay, also the data are presented in Fig. 3.

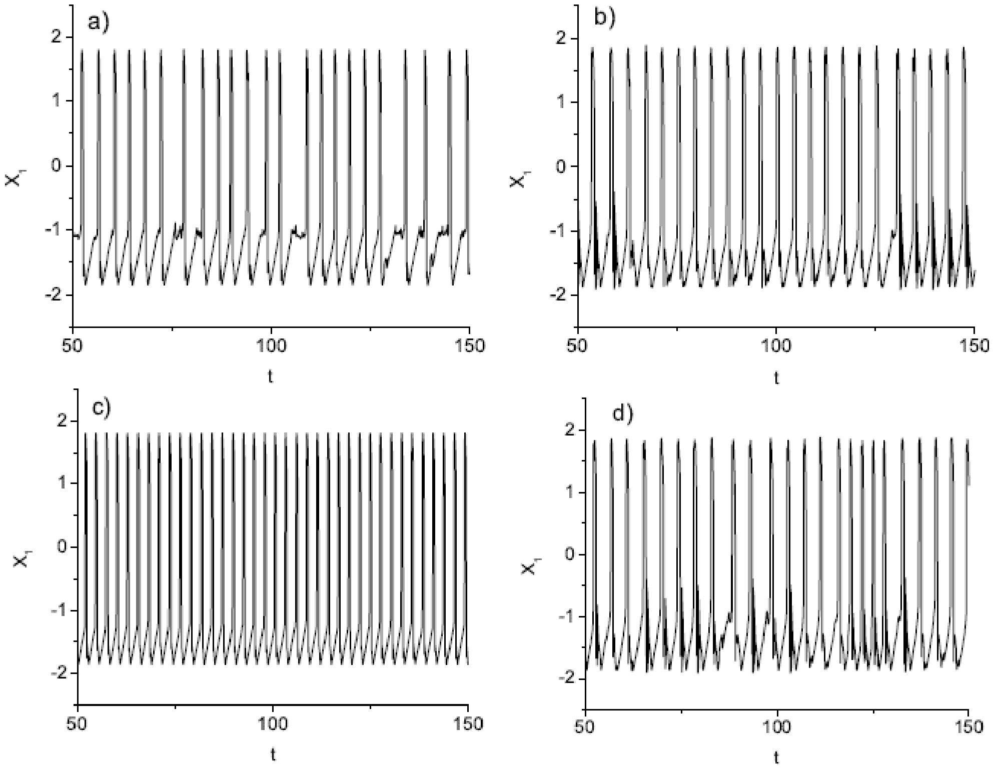

Figure 3. Time series $x_1(t)$: a) $\tau_{in} = 0$; $\tau = 0.7$, b) $\tau_{in} = 0.2$; $\tau = 0.7$; c) $\tau_{in} = 0$; $\tau = 1.3$; d) $\tau_{in} = 0.2$; $\tau = 1.3$.

If the internal delay is $\tau_{in} < \tau_{in,c}$ the coherence starts to show strong dependence on the coupling delay for values of $\tau_c \approx 1.15$, as already noticed in /18/ because of the proximity to the fold limit cycle bifurcation induced by the interaction delay. Also if $\tau_{in} > \tau_{in,c}$ and for the considered values of the coupling strength c, variation of the coupling delays $\tau \leq 1.1$ produces no significant effect on the coherence $SNR_{1,2}$. The strong effect on the coherence of the coupling delay $\tau > 1.15$ is quite different for different values of the internal delays τ_{in}. The small coherence observed when there are two stable limit cycles, one induced by $\tau > 1.15$ is a consequence of the stochastic switching between the two cycles.

CONCLUSION

We have studied a pair of type II excitable noisy systems with internal and interaction delays. The FitzHugh-Nagumo model is used as a paradigm of type II excitability. Different time-scales between the internal delay and coupling dynamics justify the interaction time-delay, and the difference in the time scale of the excitable and the refractory variables justify the internal time delays. The white noise terms have been introduced in the equation of excitatory variables, the so-called self-induced stochastic resonance (SISR) case.

Influence of instantaneous and delayed interaction of coherence spike trains displayed by each of the excitable units has been studied before. In this paper we have studied the manifestation of the interplay between the influence of the internal delays, the coupling delays, and noise intensity variations of the coherence of the spike trains elicited by the two units. We have presented most interesting results of detailed numerical computations of:

a) the dependence of the coherence properties on the small internal delays and noise intensity, and

b) variation of this dependence on the coupling delays.

The major effect of the internal delays is the improved coherence of the spike trains for a certain range of internal delays. There are relevant domains of values of the internal, and the interaction delays, such that each independently leads to significant improvement of the coherence, but jointly, the two delays imply quite incoherent oscillations.

All major effects of the two types of delays are explained by the dynamical implications and the properties of the subcritical Hopf bifurcation induced by the interaction delay. It will be interesting to study the same type of investigation in a pair of type I excitable systems, represented for example by the Terman-Wang systems.

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