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MATHEMATICAL METHOD TO DETERMINE THERMAL STRAIN RATES AND DISPLACEMENT IN A THICK-WALLED SPHERICAL SHELL

MATEMATIČKA METODA ZA ODREĐIVANJE BRZINE TOPLOTNE DEFORMACIJE I POMERANJA KOD DEBELOZIDNE SFERNE LJUSKE

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- strain rates
- displacement
- spherical shell
- stresses

Abstract

Seth's transition theory is applied to the problem of thermal creep strain rates and displacement in a thick-walled spherical shell by finite deformation. Neither the yield criterion nor the associated flow rule are assumed here. The results obtained here are applicable to compressible materials. If the additional condition of incompressibility is imposed, then the expression for stresses corresponds to those arising from Tresca yield condition. It has been observed that the circumferential stresses have maximum value at the external surface of thick wall spherical shell made of compressible materials as compared to the incompressible material applied through temperature for measure n = 0.142. Strain rates have a maximum value at the external surface for measure n = 0.142, but the result is reversed in the case of measure n = 0.2 and 0.33.

Ključne reči
- brzina deformacije
- pomeranje
- sferna ljuska
- naponi

Izvod

Teorija prelaznog stanja Seta je primenjena sa konačnim deformacijama na problem brzine puzanja i pomeranja kod debelozidne sferne ljuske. Ovde se ne pretpostavlja ni kriterijum puzanja a ni odgovarajući zakon protoka. Dobijeni rezultati se mogu primeniti na stišljive materijale. Ako bi se zadao dodatni uslov nestišljivosti, onda su izrazi za napone isti kao pri izvođenju primenom uslova tečenja Treska. Uočava se da naponi na obimskom pravcu imaju najveću vrednost na spoljnoj površini debelozidne sferne ljuske sačinjene od stišljivog materijala u poređenju sa nestišljivim materijalom pri izvođenjima temperature i pri mernoj veličini n = 0,142. Brzine deformacije imaju najveće vrednosti na spoljnoj površini kod vrednosti merne veličine n = 0,142, ali se rezultat menja u suprotnom smeru kod mernih veličina n = 0,2 i 0,33.

INTRODUCTION

The main topic of this paper is a thick-walled spherical shell of rubber, copper, or brass-like material. Therefore, studies and investigations on different axisymmetric shells are carefully reviewed and their keynotes are mentioned here. Shells are common structural elements in many engineering applications, including pressure vessels, submarine hulls, ship hulls, wings and fuselages of airplanes, missiles, automobile tires, pipes, exteriors of rockets, concrete roofs, chimneys, cooling towers, liquid storage tanks, and many other structures. They are also found in nature in the form of eggs, leaves, inner ear, bladder, blood vessels, skulls, and geological formations.

Rotating shell structures have many engineering applications like aviation, rocketry, missiles, electric motors and locomotive engines. Engineers have found its increasing application in aerospace, chemical, civil and mechanical industries such as in high-speed centrifugal separators, gas turbines for high-power aircraft engines, spinning satellite structures, certain rotor systems and rotating magnetic shields, /7/. To increase the strength of shells or shafts, it is therefore very important for engineers to study the behaviour of transition in rotating shells. A shell is a curved surface in which the thickness is much smaller compared to the other dimensions. Geometrical properties of shells, i.e. the single or double curvature give rise to a tremendous advantage of these light-weight structures, /27/. Analysis and design of these structures are, therefore, continuously of interest to the scientific and engineering community. The accurate and conservative assessments of the maximal load carried by the structure, as well as the equilibrium path in both elastic and plastic range are of paramount importance. Solutions for thin spherical shells can be found in most of the standard elasticity and plasticity textbooks /5, 9/. Elastic behaviour of shells has been very closely investigated, mostly by means of finite element method. Many authors like R. Eberlein, Wriggers, Civalek, Gürses have done elastic-plastic calculations in shells by using various theoretical and numerical approaches based on finite element method, shear deformation theory, discrete convolution technique, /5-8/. This paper is based on the non-linear transition theory of elastic-plastic shells. Here, the elastic-plastic problem of rotating spherical shells based on the different degree of compressibility has been solved by using the concept of generalized strain measures and transition theory. The distribution of stresses and yielding in an elastic-plastic rotating shell has been calculated by using the concept of generalized strain measures and generalized Hooke's law at critical points of the non-linear differential equation defining the equilibrium stage. The transition theory of elastic-plastic shells does not implement ad-hoc assumptions as incompressibility, yield conditions, those of Tresca, Von Mises, and creep-strain laws as those of Norton, Odquist, /8/. This theory has been used to solve various elastic-plastic transition problems, /3, 4, 12/. The accurate calculation of radial and circumferential stresses is essential for efficient design and long life of mechanical structures. In this paper, elastic-plastic stresses are determined by using the asymptotic solution at critical points and required angular speed to start initial yielding in the shell without using any semi-empirical yield condition and other certain laws. We analyse the non-linear transition problem of a thin rotating spherical shell by using generalized strain measures and Seth's transition theory for different values of compressibility. The effect of displacement and strain rates has been discussed numerically and is depicted graphically.

GOVERNING EQUATIONS OF THE PROBLEM

A thick-walled spherical shell, whose internal and external radii are a and b respectively, is subjected to uniform internal pressure p of gradually increasing magnitude and a temperature Θ applied to the internal surface of the shell. It is convenient to use spherical polar coordinates (r, θ, ϕ), where θ is the angle made by the radius vector with a fixed axis, and ϕ is the angle measured around this axis By virtue of the spherical symmetry $\sigma_\theta = \sigma_\phi$ everywhere in the shell, due to spherical symmetry of the structure, the components of displacement in spherical coordinates (r, θ, ϕ) are given by /13/:

$$u = r(1-\beta),\ \ v = 0,\ \ w = dz \tag{1}$$

where: u, v, w are displacement components; β is a position function, depending on $r = \sqrt{x^2+y^2+z^2}$ only. Generalized components of strain are given by Seth, /13/:

$$e_{rr} = \frac{1}{n}\left[1-(r\beta'+\beta)^n\right],\ e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right] = e_{\phi\phi},$$
$$e_{r\theta} = e_{\theta\phi} = e_{\phi r} = 0 \tag{2}$$

where: $\beta' = d\beta/dr$.

Stress-strain relation: The T stress-strain relations for thermoelastic isotropic material are, /7/:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij} - \xi\Theta\delta_{ij},\ \ (i,j = 1, 2, 3) \tag{3}$$

where: T_{ij} are stress components; λ and μ are Lame's constants; $I_1 = e_{kk}$ is the first strain invariant; δ_{ij} is the Kronecker delta; $\xi = \alpha(3\lambda + 2\mu)$; α being the coefficient of thermal expansion, and Θ is the temperature. Further, Θ has to satisfy:

$$\nabla^2\Theta = 0 \tag{4}$$

Using Eq.(2) in Eq.(3), the stresses are obtained as:

$$T_{rr} = \frac{\lambda+2\mu}{n}\left[1-(r\beta'+\beta^n)\right] + \frac{2\lambda}{n}[1-\beta^n] - \xi\Theta$$
$$T_{\theta\theta} = \frac{\lambda}{n}\left[1-(r\beta'+\beta^n)\right] + \frac{2(\lambda+\mu)}{n}[1-\beta^n] - \xi\Theta = T_{\phi\phi} \tag{5}$$
$$T_{r\theta} = T_{\theta\phi} = T_{\phi r} = 0$$

Equation of equilibrium: The radial equilibrium of an element of the rotating disk requires:

$$\frac{\partial T_{rr}}{\partial r} = \frac{2}{r}(T_{\theta\theta} - T_{rr}) \tag{6}$$

where T_{rr} and $T_{\theta\theta}$ are the radial and hoop stresses. For sufficiently small values of pressure, the deformation of the shell is purely elastic. If the radial displacement is denoted by u, the stress-strain relations for the elastic shell may be written as:

$$e_{rr} = \frac{\partial u}{\partial r} = \frac{1}{E}\left[T_{rr} - 2\nu T_{\theta\theta}\right]$$
$$e_{\theta\theta} = e_{\phi\phi} = \frac{u}{r} = \frac{1}{E}\left[(1-\nu)T_{\theta\theta} - \nu T_{rr}\right] \tag{7}$$

Boundary conditions: The temperature satisfying Laplace Eq.(4) with boundary condition:

$$\Theta = \Theta_0 \text{ and } T_{rr} = -p;\ u = 0 \text{ at } r = a$$
$$\Theta = 0 \text{ and } T_{rr} = 0 \text{ at } r = b \tag{8}$$

where Θ_0 is constant, given by /7/:

$$\Theta = \frac{\Theta_0 a(b-r)}{r(b-a)}$$

Critical points or turning points: Using Eqs.(5) and (8) in Eq.(6) we get a non-linear differential equation in β as:

$$\frac{\beta P(P+1)^{n-1}}{\left[1-\beta^n(P+1)^n\right]}\frac{dP}{d\beta} + P(P+1)^n + 2(1-C)P +$$
$$+\frac{C\xi\bar{\Theta}_0}{2\mu r\beta^n} - \frac{2C}{n\beta^n}\left[\left\{1-\beta^n(P+1)^n\right\} - (1-\beta^n)\right] = 0 \tag{9}$$

where: $\bar{\Theta}_0 = -\Theta_0 ab/(b-a)$; $C = 2\mu/\lambda + 2\mu$ and $r\beta' = \beta P$ (P is function of β and β is function of r).

Transition points of β in Eq.(9) are $P \to -1$ and $P \to \pm\infty$.

SOLUTION OF THE PROBLEM

To find thermal creep stresses and strain rates, the transition function is taken through principal stress difference /11, 14, 17-26/ at the transition point $P \to -1$. We define the transition function γ as:

$$\gamma = T_{rr} - T_{\theta\theta} = \frac{2\mu\beta^n}{n}\left[\left\{1-\beta^n(P+1)^n\right\} - (1-\beta^n)\right] \tag{10}$$

where γ is a function of r only and γ is the dimension.

Taking the logarithm and differentiation of Eq.(10) with respect to r and substituting the value of $dP/d\beta$ from Eq.(9) and taking asymptotic value $P \to -1$, after integration we get:

$$\gamma = T_{rr} - T_{\theta\theta} = A_0 r^{-2c}\left[1-(1-\beta^n)\right]^{3-2C}\exp(F_1) \quad (11)$$

where: $C\xi = \sigma(3-2C)$; $E = 2\mu(3-2C)/(2-C)$; $F_1 = \alpha(3-2C)\bar{\Theta}_0 \int \frac{dr}{r^2\left[1-(1-\beta^n)\right]}$; A_1 and F_0 are constants of integration, that can be determined by boundary condition. The asymptotic value of β as $P \to -1$ is D/r, D being a constant, therefore from Eq. (11), we have

$$\gamma = T_{rr} - T_{\theta\theta} = A_0 r^{-2C}\left[D^n r^{-n}\right]^{3-2C}\exp(F_2) \quad (12)$$

where $F_2 = \alpha(3-2C)\bar{\Theta}_0 \int \frac{dr}{r^2\left[1-(1-D^n r^{-n})\right]}$.

Substituting Eq.(12) into Eq.(6), we get:

$$T_{rr} = -2A_0 \int r^{-2C-1}[D^n r^{-n}]^{3-2C}\exp(F_2)dr + A_1 \quad (13)$$

where: A_1 is a constant of integration that can be determined by boundary condition. Using boundary condition Eq.(8) in Eq.(13), we get $A_1 = [2A_0 \int r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr]_{r=b}$.

Substituting the constants A_1 in Eq.(13), we get:

$$A_0 = \frac{-p}{2\int_a^b r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr}.$$

Substituting the value of the constants A_0 and A_1 in Eqs.(12), (13) and (7), we get:

$$T_{rr} = -p\frac{\int_r^b r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr}{\int_a^b r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr} \quad (14)$$

$$T_{\theta\theta} = T_{\phi\phi} = T_{rr} + \frac{pr^{-2C}\{D^n r^{-n}\}^{3-2C}\exp(F_2)}{2\int_a^b r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr} \quad (15)$$

$$u = \frac{p}{\int_a^b r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr}\left[\frac{(1-\nu)r^{-2C}\{D^n r^{-n}\}^{3-2C}\exp(F_2)}{2} - (1-2\nu)\int_r^b r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr\right] \quad (16)$$

where: $F_2 = \frac{\alpha\bar{\Theta}_0(3-2C)r^{n-1}}{(n-1)D^n}$; α is the coefficient of thermal expansion. Eqs. (14)-(16) define creep stresses and displacement for a thick spherical shell under uniform pressure. We introduce now the following non-dimensional components: $R = r/b$, $R_0 = a/b$, $\sigma_r = T_{rr}/p$, $\sigma_\theta = T_{\theta\theta}/p$, $D = 1$, $\bar{u} = u/p$, and $\alpha\Theta_0 = \Theta_1$, to get Eqs. (14)-(26) in non-dimensional form:

$$\sigma_r = -\frac{\int_R^1 R^{-3n+[2C(n-1)-1]}\exp(F_2)dR}{\int_{R_0}^1 R^{-3n+[2C(n-1)-1]}\exp(F_2)dR} \quad (17)$$

$$\sigma_\theta = \sigma_\phi = \sigma_r + \frac{R^{-3n+2C(n-1)}\exp(F_2)}{2\int_{R_0}^1 R^{-3n+2C(n-1)}\exp(F_2)dR} \quad (18)$$

$$u = \frac{p}{\int_a^b r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr} \times \left[\frac{(1-\nu)r^{-2C}\{D^n r^{-n}\}^{3-2C}\exp(F_2)}{2} - (1-2\nu)\int_r^b r^{-2C-1}\{D^n r^{-n}\}^{3-2C}\exp(F_2)dr\right] \quad (19)$$

where: $F_2 = \frac{\Theta_1(3-2C)R_0(1-R)R^{n-1}b^{n-1}}{(1-R_0)(n-1)}$.

ESTIMATION OF CREEP PARAMETERS

When creep sets in, the strains should be replaced by strain rates and the stress-strain relations Eq.(3) become:

$$\dot{e}_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}T + \alpha\Theta \quad (20)$$

where $\dot{e}_{ij}$ is the strain rate tensor with respect to flow parameter t. Differentiating Eq.(4) with respect to time, we get:

$$\dot{e}_{\theta\theta} = -\beta^{n-1}\dot{\beta} \quad (21)$$

For SWAINGER measure (i.e. $n = 1$), Eq.(21) becomes:

$$\dot{\varepsilon}_{\theta\theta} = \dot{\beta} \quad (22)$$

where $\dot{\varepsilon}_{\theta\theta}$ is the SWAINGER strain measure. From Eq. (10) the transition value β is given by:

$$\beta = (n/2\mu)^{1/n}[\sigma_{rr} - \sigma_{\theta\theta}]^{1/n} \quad (23)$$

Using Eqs.(21)-(23) in Eq.(24), we get:

$$\dot{\varepsilon}_{rr} = \left[n(\sigma_r - \sigma_\theta)(1+\nu)\right]^{\frac{1}{n}-1}[\sigma_r - \nu\sigma_\theta + \alpha\Theta]$$

$$\dot{\varepsilon}_{\theta\theta} = \left[n(\sigma_r - \sigma_\theta)(1+\nu)\right]^{\frac{1}{n}-1}[\sigma_\theta - \nu\sigma_r + \alpha\Theta] \quad (24)$$

$$\dot{\varepsilon}_{zz} = -\left[n(\sigma_r - \sigma_\theta)(1+\nu)\right]^{\frac{1}{n}-1}\left[\nu(\sigma_r + \sigma_\theta) + \alpha\Theta\right]$$

where: $\dot{\varepsilon}_{rr}$, $\dot{\varepsilon}_{\theta\theta}$ and $\dot{\varepsilon}_{zz}$ are strain rate tensor. These are the constitutive equations used by Odquist /8/ for finding the creep stresses and strain rates provided we put $n = 1/N$.

NUMERICAL RESULTS DISCUSSION

For calculating strain rates, stresses and displacement based on the above analysis, the following values have been taken $\nu = 0.5$ (incompressible material, i.e. rubber), $\nu = 0.4285$ (compressible material, i.e. saturated clay), and $\nu = 0.333$ (compressible materials, i.e. copper), $n = 1/3$, 1/5, 1/7 (i.e $N = 3, 5, 7$), $\alpha = 5.0 \times 10^{-5}$ °F^{-1} (for methyl methacrylate, /8/, $\Theta_1 = 0$, 0.125 and $D = 1$. In classical theory measure, N equals to $1/n$.

Definite integrals in Eqs.(17)-(18) have been solved by using Simpson's rule. Curves are produced between stresses along the radii ratio $R = r/b$ (see Fig. 2(a)) for thick-walled spherical shell made of compressible as well as incomepressible material with temperature $\Theta_1 = 0$, 0.125 and measure $n = 0.142$, 0.2, and 0.333. It is also observed from Fig. 2 that the circumferential stresses have maximum value at the external surface of thick-walled spherical shell made of compressible material as compared to the incompressible material applied through temperature $\Theta_1 = 0.125$ for measure $n = 0.142$. But the result is reversed in measure $n = 0.2$ and 0.33. For measure $n = 0.2$ and 0.33, it has been seen that circumferential stresses are maximum at the internal surface of the compressible material, with the introduction of the thermal effect, decrease the values of stresses at the internal surface.

Curves are produced for strain rates along the radii ratio $R = r/b$ (see Fig. 3) for thick-walled spherical shell of compressible material (i.e. saturated clay or copper) as well as incompressible material (i.e. rubber) with temperature $\Theta_1 = 0$, 0.125 and measure $n = 1/7$, 1/5, 1/3 (i.e. $N = 7, 5, 3$). It has been seen (Fig. 3) that the thick-walled spherical shell of compressible material has a maximum value of strain at the external surface as compared to the shell of incompressible material for measure $n = 0.142$ at $\Theta_1 = 0.125$. But a reversed result is obtained for measure $n = 0.2$ and 0.33.

CONCLUSION

It has been observed that circumferential stresses have maximum value at the external surface of a thick-walled spherical shell of compressible material as compared to an incompressible material applied through temperature for measure $n = 0.142$. But a reverse result is reached in measure $n = 0.2$ and 0.33. Strain rates have a maximum value at the external surface for measure $n = 0.142$, but a reversed result is received in the case of measure $n = 0.2$ and 0.33.

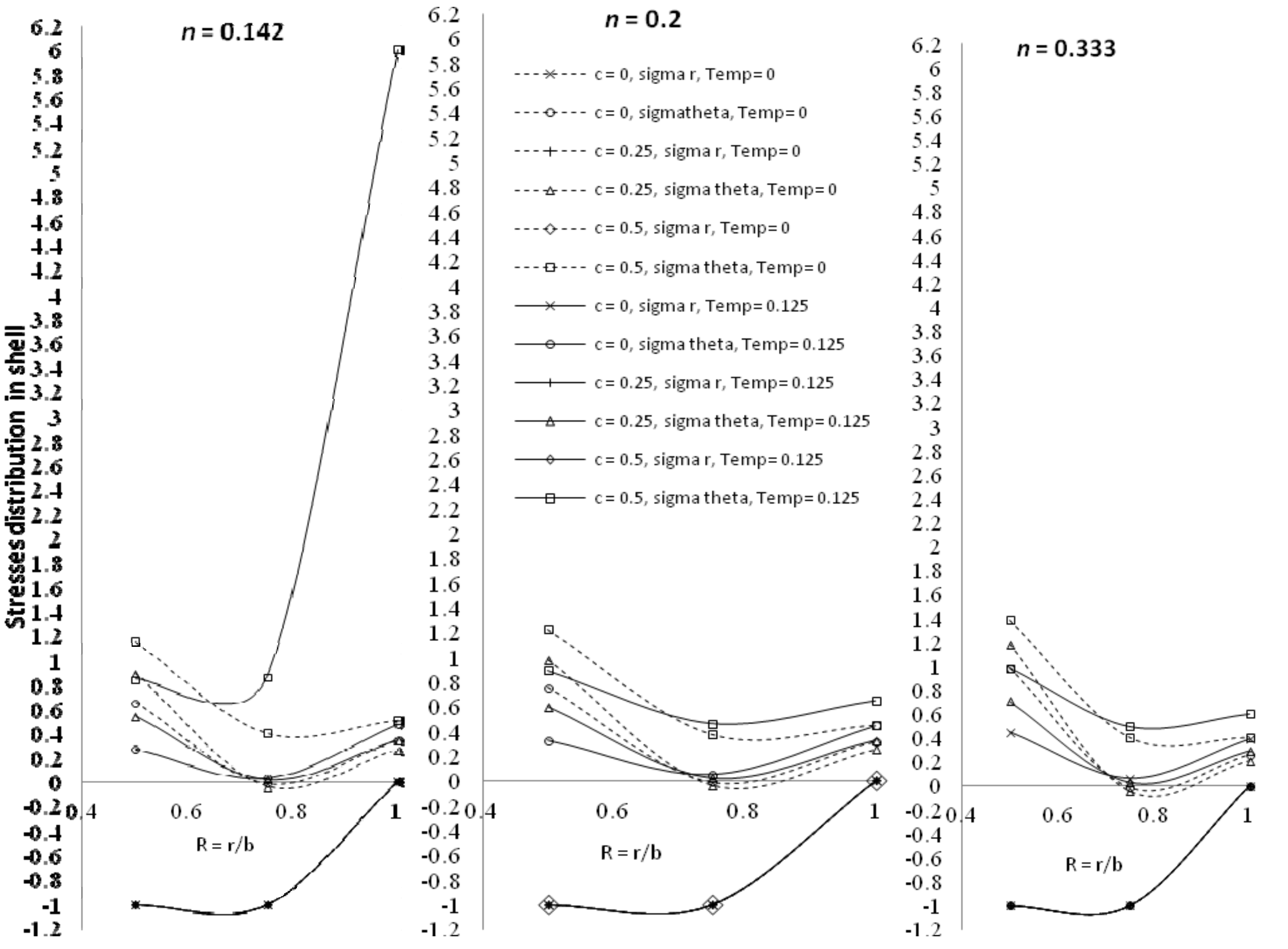

Figure 1. Stress distribution in a thick-walled spherical shell along radius $R = r/b$.

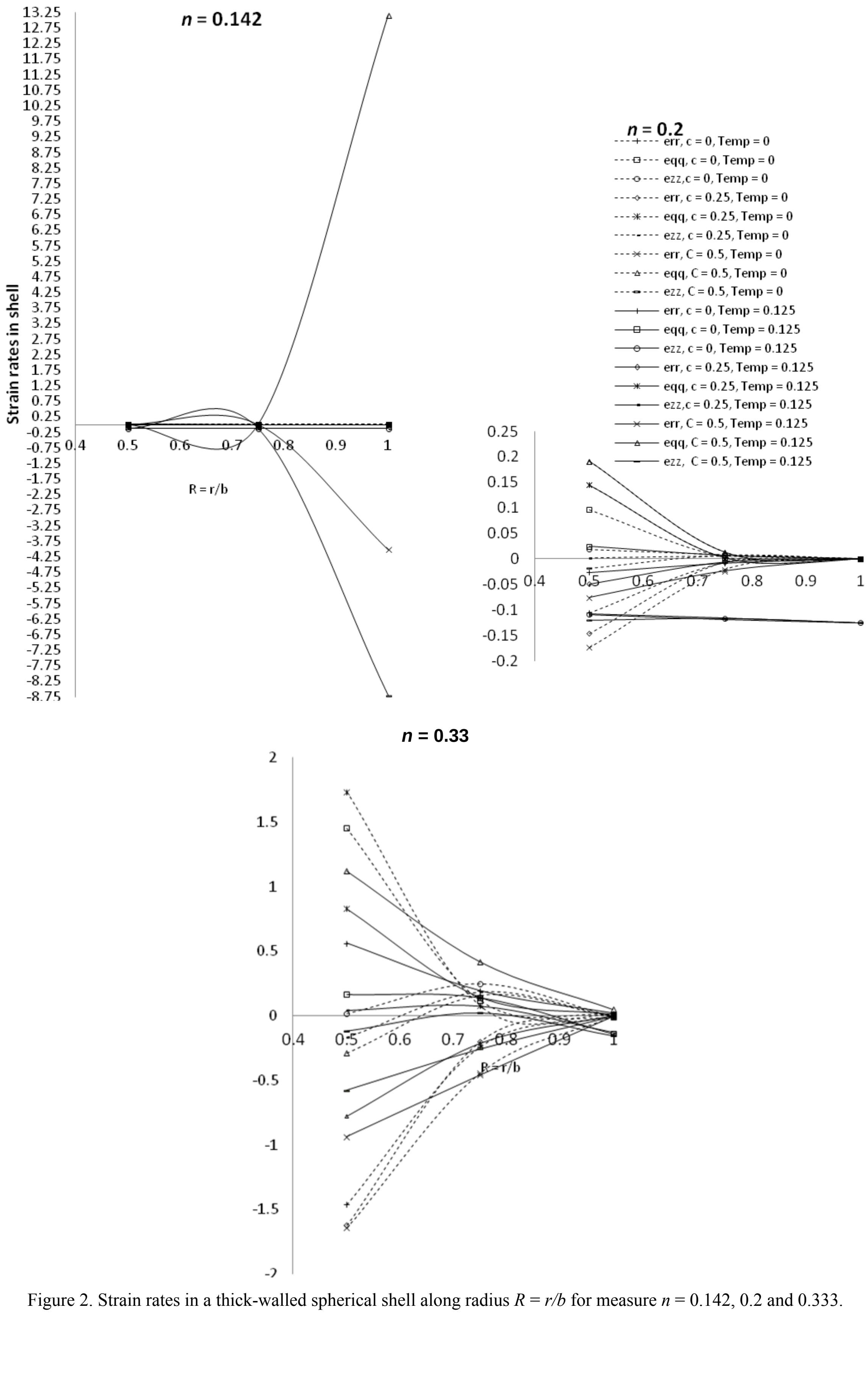

Figure 2. Strain rates in a thick-walled spherical shell along radius $R = r/b$ for measure n = 0.142, 0.2 and 0.333.

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