Thakur Pankaj[1], Satya Bir Singh[2], Jasmina Lozanović Šajić[3]

THERMO ELASTIC-PLASTIC DEFORMATION IN A SOLID DISK WITH HEAT GENERATION SUBJECTED TO PRESSURE

TERMO-ELASTOPLASTIČNA DEFORMACIJA ČVRSTOG DISKA SA IZVOROM TOPLOTE OPTEREĆENIM NA PRITISAK

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Adresa autora / Author's address:
1) Depart. of Mathematics, IEC University Baddi, Solan, Himachal Pradesh, India pankaj_thakur15@yahoo.co.in
2) Depart. of Mathematics, Punjabi University Patiala, Punjab, India
3) Innov. Centre, Faculty of Mech. Engng., University of Belgrade, Serbia

Keywords
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Abstract

Seth's transition theory is applied to the problem of elastic-plastic deformation in a solid disk due to heat source and subjected to pressure. Neither the yield criterion nor the associated flow rule is assumed here. The results obtained here are applicable to compressible materials. If the additional condition of incompressibility is imposed, then the expression for stresses corresponds to those arising from Tresca's yield condition. The present solution is illustrated by numerical results and is compared with uniform heat generation case. This work provides the basis for a comprehensive investigation of the influence of non-uniform heat generation subjected to pressure. Effect of heat increased values of stress for compressible material at the inner surface.

Ključne reči
- čvrsti disk
- pritisak
- toplota
- naponi
- tečenje
- pomeranje

Izvod

Setova teorija prelaznog stanja je primenjena na problem elastoplastičnog deformisanja čvrstog diska pod uticajem izvora toplote i pri opterećenju na pritisak. U ovom slučaju nije pretpostavljen kriterijum tečenja, a niti odgovarajući zakon protoka. Dobijeni rezultati su primenljivi na stišljiv materijal. Ako bi se zadao dodatni uslov nestišljivosti, onda izrazi za napone odgovaraju istim onim koji se izvode primenom uslova tečenja Treska. Dato rešenje je ilustrovano numeričkim rezultatima i dato je poređenje sa slučajem kada postoji uniformni izvor toplote. Rad pruža osnove za sveobuhvatno istraživanje uticaja promenljivog izvora toplote uz dejstvo pritisnog naprezanja. Uticaj toplote dovodi do povećanja napona u stišljivom materijalu na njegovoj unutrašnjoj površini.

INTRODUCTION

The use of rotating disks in machinery and structural applications has generated considerable interest in many problems in the domain of solid mechanics. There are many applications of such type of rotating disks, such as high speed gears, turbine rotors, flywheels, disk drives, etc. The stress distribution in an elastic-plastic rotating solid disk was first introduced by F. Laszlo, /1/. The usual approach for the determination of the elastic-plastic stress distribution is to apply the principle of momentum, Hooke's law, Tresca's yield condition and the condition of vanishing of radial stress at the outer surface of the disk. The first modern treatment for the elastic-plastic annular and solid disk with linear strain hardening has been given by Gamer, /2-4/. It was shown by Gamer /2/ that the analysis based on Tresca's yield condition for the elastic-perfectly plastic rotating solid disk is not meaningful. Accordingly, it was shown by Gamer /3/ that a meaningful solution for linear strain hardening can be obtained. In the analysis of Gamer, the plastic region of the disk in the elastic-plastic state consists of two plastic regimes with different forms of the yield condition, the inner being a corner regime and the outer a side regime of Tresca's hexagon. Thakur et al. /5/ investigated infinitesimal deformation in a solid disk by using Seth's transition theory. You et al. /6/ analysed elastic-plastic stresses in a rotating solid disk. Ahmet N., Eraslan et al. /8/ analysed rotating elastic-plastic solid disks of variable thickness having concave profiles. Seth's transition theory, /8/, does not acquire any assumptions like a yield condition, incompressibility condition, and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. It utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deforming field, and has been successfully applied to a large number of problems /5, 8-12/. Seth /8/ has defined the generalized principal strain measure as:

$$e_{ii} = \int_0^{\overset{A}{e}_{ii}} \left[1 - 2\overset{A}{e}_{ii}\right]^{\frac{n}{2}-1} d\overset{A}{e}_{ii} = \frac{1}{n}\left[1 - \left(1 - 2\overset{A}{e}_{ii}\right)\right],\ i = 1,2,3 \quad (1)$$

where *'n'* is the measure.

In this paper, thermo elastic-plastic stress analysis of a rotating solid disc with thermal effect subjected to internal pressure is presented. Results are discussed numerically and depicted graphically. In general, steady-state temperature heat generation may be a function of space or temperature, /10/. This work is concerned with the elastic-plastic stress analysis in a finitesimal deformation of solid disk with effect of heat generation. For the problem considered here, heat generation rate is a function of the radial position in the form

$$\dot{q}(r) = q_0\left[1 - \left(\frac{r}{a}\right)^s\right] \quad (2)$$

where q_0 is the magnitude of the heat generation at $r = 0$, r is measured from the centre of solid disk, a is the radius of the disk. The analysis is based on usual assumptions of a plane stress state. Results have been discussed numerically and depicted graphically.

Mathematical model of solid disk

We consider a state of plane stress and assume infinitesimal deformation. Suppose a solid disk having constant density with radius b. The disk is rotating with angular velocity ω about an axis perpendicular to its plane and passing through the centre as shown in Fig. 1. The thickness of the disk is assumed to be constant and is taken to be sufficiently small so that the disk is effectively in a state of plane stress, thus the axial stress T_{zz} is zero.

GOVERNING EQUATIONS

The cylindrical polar coordinates are given by Seth /9/:

$$u = r(1-\beta);\quad v = 0 \ \text{and}\ w = dz \quad (3)$$

where β is function of $r = \sqrt{x^2+y^2}$ only, d is a constant.

The strain components for infinitesimal deformation are:

$$\begin{aligned}
\overset{A}{e}_{rr} &\equiv \frac{\partial u}{\partial r} = \frac{1}{2}\left[1 - (\beta + r\beta')\right]\\
\overset{A}{e}_{\theta\theta} &\equiv \frac{u}{r} = \frac{1}{2}\left[1-\beta\right]\\
\overset{A}{e}_{zz} &\equiv \frac{\partial w}{\partial z} = d\\
\overset{A}{e}_{r\theta} &= \overset{A}{e}_{\theta z} = \overset{A}{e}_{zr} = 0
\end{aligned} \quad (4)$$

Using Eq.(4) in Eq.(1), the generalised components of strain are:

$$\begin{aligned}
e_{rr} &= \frac{1}{n}\left[1 - \left\{2(\beta + r\beta') - 1\right\}^{n/2}\right]\\
e_{\theta\theta} &= \frac{1}{n}\left[1 - \left\{2\beta - 1\right\}^{n/2}\right]\\
e_{zz} &= \frac{1}{n}\left[1 - (1-2d)^{n/2}\right]\\
e_{r\theta} &= e_{\theta z} = e_{zr} = 0
\end{aligned} \quad (5)$$

where $\beta' = d\beta/dr$.

Stress-strain relations for isotropic media is given by /10/:

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu e_{ij} - \xi\Theta\delta_{ij},\ i,j = 1,2,3 \quad (6)$$

where T_{ij} are the stress components; e_{ij} are the strain components; λ, μ are Lame's constants; $I_1 = e_{kk}$ is the first strain invariant; δ_{ij} is the Kronecker's delta; and $\xi = \alpha(3\lambda + 2\mu)$; α being the coefficient of thermal expansion; and Θ is temperature. Further, Θ has to satisfy the heat equation which gives /10/:

$$\begin{aligned}
&\nabla^2\Theta = 0\\
&\frac{1}{r}\frac{d}{dr}\left[r\frac{d\Theta(r)}{dr}\right] + \frac{\dot{q}(r)}{k} = 0,\ \text{in } 0 < r < a
\end{aligned} \quad (7)$$

The temperature field satisfying Eq.(7) and $d\Theta(r)/dr = 0$ at $r = 0$, and $\Theta(r) = 0$ at $r = a$.

Using Eq.(2) and these boundary conditions, the steady-state temperature distribution is obtained as

$$\Theta(r) = \frac{q_0 a^2}{4k}\left[1 - \left(\frac{r}{a}\right)^2 - \frac{4}{(s+2)^2}\left[1 - \left(\frac{r}{a}\right)^{s+2}\right]\right]$$

where k is the thermal conductivity. Equation (8) for this problem becomes:

$$\begin{aligned}
T_{rr} &= \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr} + e_{\theta\theta}\right] + 2\mu e_{rr} - \frac{2\mu\xi\Theta}{\lambda+2\mu}\\
T_{\theta\theta} &= \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr} + e_{\theta\theta}\right] + 2\mu e_{\theta\theta} - \frac{2\mu\xi\Theta}{\lambda+2\mu}\\
T_{zz} &= T_{zr} = T_{r\theta} = T_{\theta z} = 0
\end{aligned} \quad (8)$$

From Eq.(8), strain components in terms of stresses are obtained as:

$$\begin{aligned}
e_{rr} &= \frac{\partial u}{\partial r} = \frac{1}{E}\left[T_{rr} - \nu T_{\theta\theta}\right] + \alpha\Theta\\
e_{\theta\theta} &= \frac{u}{r} = \frac{1}{E}\left[T_{\theta\theta} - \nu T_{rr}\right] + \alpha\Theta\\
e_{zz} &= -\frac{\nu}{E}\left[T_{rr} - T_{\theta\theta}\right] + \alpha\Theta\\
e_{rr} &= e_{\theta z} = e_{zr} = 0
\end{aligned} \quad (9)$$

where $E = \frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$ and $\nu = \frac{\lambda}{2(\lambda+\mu)}$. Substituting Eq.(5) in Eq.(6), the stresses are obtained as:

$$\begin{aligned}
T_{rr} &= \frac{2\mu}{n}\Big[3 - 2C - \left\{2\beta(P+1) - 1\right\}^{n/2}(2-C) - \\
&\quad -(2\beta-1)^{n/2}(1-C) - \frac{nC\xi\Theta}{2\mu}\Big]\\
T_{\theta\theta} &= \frac{2\mu}{n}\Big[3 - 2C - \left\{2\beta(P+1) - 1\right\}^{n/2}(1-C) - \\
&\quad -(2\beta-1)^{n/2}(2-C) - \frac{nC\xi\Theta}{2\mu}\Big]\\
T_{zz} &= T_{zr} = T_{r\theta} = T_{\theta z} = 0
\end{aligned} \quad (10)$$

where $r\beta' = \beta P$ and $C = \frac{2\mu}{\lambda+2\mu}$. The equations of equilibrium are all satisfied except:

$$\frac{d}{dr}(rT_{rr})-T_{\theta\theta}+\rho\omega^2r^2=0\,. \tag{11}$$

Using Eqs.(10) and (7) in Eq.(10), we get a non-linear differential equation in β as:

$$(2-C)n\beta^2P\{2\beta(P+1)-1\}^{\left(\frac{n}{2}-1\right)}\frac{dP}{d\beta}=\left[\left\{\frac{n\rho\omega^2r^2}{2\mu}-\right.\right.$$
$$-\{2\beta(P+1)-1\}^{n/2}\left[1+\frac{n\beta P(P+1)(2-C)}{\{2\beta(P+1)-1\}}\right]+\{2\beta-1\}^{n/2}\times$$
$$\left.\left.\times\left[1-\frac{n\beta P(1-C)}{\{2\beta-1\}}\right]\right\}-\frac{nC\xi q_0r^2}{8\mu k}\left[\frac{4}{(s+2)}\left(\frac{r}{a}\right)^s-\frac{2}{a}\right]\right] \tag{12}$$

where $\Theta'(r)=\frac{q_0a^2}{4k}\left[\frac{4}{(s+2)}\left(\frac{r}{a}\right)^{s+1}-\frac{2}{a}\right]$.

Transition points: Transition points of β in Eq.(12) are $P\to 0$ and $P\to\pm\infty$. $P\to 0$ gives nothing of importance.

Boundary condition: The boundary conditions are:

$$u=0 \text{ at } r=0 \text{ and } T_{rr}=-p \text{ at } r=b. \tag{13}$$

Elastic-plastic solution of the problem

It has been shown (Seth /8, 9/, Gupta et al. /11/, Thakur /5, 12-19/) that the asymptotic solution through the principal stress leads from elastic state to plastic state at the transition point. We define the transition function R as:

$$\varsigma=\frac{n}{2\mu}\left[T_{\theta\theta}+C\xi\Theta\right]=\left[(3-2C)-\right.$$
$$\left.-\{2\beta(P+1)-1\}^{n/2}(1-C)-(2\beta-1)^{n/2}(2-C)\right] \tag{14}$$

Taking the logarithmic differentiation of Eq.(14) with respect to r and using Eq.(12), we get:

$$\frac{d(\log\varsigma)}{dr}=-\frac{\left(\frac{1-C}{2-C}\right)\left[\frac{n\rho\omega^2r^2}{2\mu\beta^n}-\{2\beta(P+1)\}^{n/2}+\right.}{r\left[3-2C-\{2\beta(P+1)-1\}^{n/2}(1-C)-(2\beta-1)^{n/2}(2-C)\right]}$$
$$\frac{\left.+(2\beta-1)^{n/2}-n\beta P(1-C)(2\beta-1)^{n/2}-\frac{nC\xi(d\Theta/dr)}{2\mu\beta^n}\right]+}{r\left[3-2C-\{2\beta(P+1)-1\}^{n/2}(1-C)-(2\beta-1)^{n/2}(2-C)\right]}$$
$$\frac{+n\beta P(2-C)(2\beta-1)^{n/2}}{r\left[3-2C-\{2\beta(P+1)-1\}^{n/2}(1-C)-(2\beta-1)^{n/2}(2-C)\right]} \tag{15}$$

Taking the asymptotic value of Eq.(15) as $P\to\pm\infty$ and after integration we get:

$$\varsigma=A_1r^{-1/(2-C)} \tag{16}$$

where A_1 is a constant of integration and can be determined by boundary condition. From Eqs.(14) and (16), we have:

$$T_{\theta\theta}=\left(\frac{2\mu}{n}\right)A_1r^{-1/(2-C)}-\frac{C\xi q_0a^2}{4k}\times$$
$$\times\left[1-\left(\frac{r}{a}\right)^2-\frac{4}{(s+2)^2}\left[1-\left(\frac{r}{a}\right)^{s+2}\right]\right] \tag{17}$$

Substituting (17) in Eq.(11) and integrating, we get:

$$T_{rr}=\left\{\frac{2\mu(2-C)}{n(1-C)}\right\}A_1r^{-1/(2-C)}-\frac{C\xi q_0a^2}{4k}\times$$
$$\times\left[1-\left(\frac{r}{a}\right)^2-\frac{4}{(s+2)^2}\left[1-\left(\frac{1}{s+3}\right)\left(\frac{r}{a}\right)^{s+2}\right]\right]-\frac{\rho\omega^2r^2}{3}+\frac{B_1}{r} \tag{18}$$

where B_1 is a constant of integration which can be determined by boundary condition. Substituting Eqs.(17) and (18) in Eq.(9), we get:

$$\frac{\partial u}{\partial r}=\frac{1}{E}\left[\left(\frac{2\mu}{n}\right)A_1r^{-1/(2-C)}\left\{\frac{3-2C}{(1-C)(2-C)}\right\}+\right.$$
$$\left.+\frac{\alpha Eq_0a^2(2-C)}{k(s+2)(s+3)}\left(\frac{r}{a}\right)^{s+2}-\frac{\rho\omega^2r^2}{3}+\frac{B_1}{r}\right] \tag{19}$$

$$\frac{u}{r}=\frac{(1-C)}{E(2-C)}\left[\frac{\rho\omega^2r^2}{3}-\frac{\alpha Eq_0a^2(1-C)}{2k}\times\right.$$
$$\left.\times\left[-1+\left(\frac{r}{a}\right)^2-\frac{2(s+4)}{(s+2)^2(s+3)}\left(\frac{r}{a}\right)^{s+2}\right]-\frac{B_1}{r}\right] \tag{20}$$

where $E=\frac{2\mu(3-2C)}{(2-C)}$ is the Young's modulus. Integrating Eq.(19) with respect to r, we get:

$$u=\frac{1}{E}\left[\left(\frac{2\mu}{n}\right)A_1r^{\frac{1-C}{2-C}}\left\{\frac{(3-2C)}{(1-C)^2}\right\}+\right.$$
$$\left.+\frac{\alpha Eq_0a^2(2-C)}{k(s+2)(s+3)^2}\left(\frac{r}{a}\right)^{s+3}-\frac{\rho\omega^2r^3}{9}+B_1\log r\right]+D \tag{21}$$

where D is a constant of integration which can be determined by boundary condition. Comparing Eqs.(20) and (21), one gets:

$$\left(\frac{2\mu}{n}\right)A_1r^{\frac{1-C}{2-C}}\left\{\frac{3-2C}{(1-C)^2}\right\}=\frac{\rho\omega^2r^3}{9}\left(\frac{5-4C}{2-C}\right)-\frac{\alpha Eq_0a^2}{k}\times$$
$$\times\left[\frac{(2-C)a}{(s+2)(s+3)^2}\left(\frac{r}{a}\right)^{s+3}+\frac{r(1-C)^2}{2(2-C)}\left\{\left(\frac{r}{a}\right)^2-1-\frac{2(s+4)}{(s+2)^2(s+3)}\left(\frac{r}{a}\right)^{s+2}\right\}\right]-$$
$$-B_1\left\{\frac{(1-C)+(2-C)\log r}{(2-C)}\right\}-DE \tag{22}$$

and

$$u=\frac{(1-C)}{E(2-C)}\left[\frac{\rho\omega^2r^3}{3}-\frac{\alpha Eq_0a^2(1-C)r}{2k}\times\right.$$
$$\left.\times\left\{-1+\left(\frac{r}{a}\right)^2-\frac{2(s+4)}{(s+2)^2(s+3)}\left(\frac{r}{a}\right)^{s+2}\right\}\right] \tag{23}$$

Using boundary conditions (13) in Eq.(23), we get: $B_1 = 0$. Putting Eq.(22) in Eq.(18) and using boundary condition (13), we get:

$$D=\frac{1}{E}\left[\frac{(3-2C)}{(2-C)(1-C)}ap+\frac{\rho\omega^2a^3}{9}\left(\frac{5-4C}{2-C}\right)-\right.$$
$$-\frac{\rho\omega^2a^3}{3}\frac{(3-2C)}{(1-C)(2-C)}-\frac{\alpha Eq_0a^3}{k(s+3)(s+3)}\times$$
$$\left.\times\left\{\left(\frac{2-C}{s+3}\right)-\frac{(1-C)^2(s+4)}{(2-C)(s+2)}+\frac{3-2C}{1-C}\right\}\right]$$

Putting values of B_1, D and using Eq.(22) in Eqs.(17) and (18) respectively, we get the plastic stresses and displacement as:

$$rT_{rr}=\left[\frac{\rho\omega^2}{9}(r^3-a^3)\frac{(1-C)(5-4C)}{(3-2C)}-\frac{\rho\omega^2}{3}(r^3-a^3)-\right.$$
$$-ap+\frac{\alpha Eq_0a^3(1-C)}{k(s+2)(s+3)}\left\{\left(\frac{r}{a}\right)^{s+3}-1\right\}\cdot\left\{\frac{(1-C)^2(s+4)}{(s+2)(3-2C)}-\frac{(2-C)^2}{(s+3)}\right\}-$$
$$-\frac{\alpha Eq_0a^2(2-C)}{2k}\left\{\left(\frac{r}{a}\right)^2-1\right\}\left\{\frac{a}{2}+\frac{r(1-C)^3}{(3-2C)}\right\}-\frac{\alpha Eq_0a^3(2-C)}{k(s+2)(s+3)}\times$$
$$\left.\times\left\{1-\frac{(s+3)}{s+2}+\frac{1}{(s+2)}\left(\frac{r}{a}\right)^{s+2}\right\}\right]$$

$$rT_{\theta\theta}=\left[\frac{\rho\omega^2}{9}(r^3-a^3)\frac{(1-C)^2(5-4C)}{(3-2C)(2-C)}-\frac{\rho\omega^2a^3(1-C)}{3(2-C)}-\right.$$
$$-\frac{\alpha Eq_0a^3(1-C)^2(2-C)}{k(s+2)(s+3)^2(3-2C)}\left\{\left(\frac{r}{a}\right)^{s+3}-1\right\}+\frac{\alpha Eq_0a^2r}{2k}\left\{\left(\frac{r}{a}\right)^2-1\right\}\times$$
$$\times\left\{\frac{2-C}{2}-\frac{(1-C)^2}{(2-C)(3-2C)}\right\}-\frac{(1-C)}{(2-C)}ap+\frac{\alpha Eq_0a^2r}{k}\left\{\left(\frac{r}{a}\right)^2-1\right\}\times$$
$$\left.\times\left\{\frac{(1-C)^4(s+4)}{(3-2C)(s+2)^2(s+3)(2-C)}-\frac{(2-C)}{(s+2)^2}\right\}+\frac{\alpha Eq_0a^3(1-C)}{k(s+2)(s+3)}\right]$$

and

$$u=\frac{(1-C)}{E(2-C)}\left[\frac{\rho\omega^2r^3}{3}-\frac{\alpha Eq_0a^2(1-C)r}{2k}\times\right.$$
$$\left.\times\left\{-1+\left(\frac{r}{a}\right)^2-\frac{2(s+4)}{(s+2)^2(s+3)}\left(\frac{r}{a}\right)^{s+2}\right\}\right] \tag{24}$$

Initial yielding: For a solid disk the stress at the centre is given when $r = 0$. For a solid disk the stress at the centre is given when $r = 0$. With r equal to zero the above Eqs.(24) will yield infinite stresses whatever the speed of rotation, these stresses are not meaningful. In this analysis an elastic solution is considered to gain *a priori* knowledge of possible plastic and elastic zones. According to the elastic solution a sketch of the general case is given in Fig. 1. Thus, the following zones can be distinguished:

- Inner plastic zone, T_{zz} (= 0) < $T_{\theta\theta}$ < T_{rr}; $0 < r \le r_1$
- Intermediate plastic zone, T_{zz} (= 0), $T_{rr} = T_{\theta\theta}$; $r_1 < r < r_2$
- Outer plastic zone, T_{zz} (= 0) < T_{rr} < $T_{\theta\theta}$; $r_2 < r < a$.

Inner plastic zone, T_{zz} (= 0) < $T_{\theta\theta}$ < T_{rr}; $0 < r \le r_1$: Yielding begins at the centre, $r = 0$, and compressive radial stress T_{rr} is greater than compressive circumferential stress $T_{\theta\theta}$. It has been seen that $|T_{rr}|$ has a maximum value at inner surface $r \le r_1$ and $r > 0$. For yielding at $r \le r_1$, Eq.(24) becomes:

$$\left|rT_{rr}\right|_{r=r_1=0.47}=\left[\frac{\rho\omega^2}{9}(r_1^3-a^3)\frac{(1-C)(5-4C)}{(3-2C)}-\frac{\rho\omega^2}{3}(r_1^3-a^3)-\right.$$
$$-ap+\frac{\alpha Eq_0a^3(1-C)}{k(s+2)(s+3)}\left\{\left(\frac{r_1}{a}\right)^{s+3}-1\right\}\left\{\frac{(1-C)^2(s+4)}{(s+2)(3-2C)}-\frac{(2-C)^2}{(s+3)}\right\}-$$
$$-\frac{\alpha Eq_0a^2(2-C)}{2k}\left\{\left(\frac{r_1}{a}\right)^2-1\right\}\left\{\frac{a}{2}+\frac{r_1(1-C)^3}{(3-2C)}\right\}-\frac{\alpha Eq_0a^3(2-C)}{k(s+2)(s+3)}\times$$
$$\left.\times\left\{1-\frac{(s+3)}{s+2}+\frac{1}{(s+2)}\left(\frac{r_1}{a}\right)^{s+2}\right\}\right]=Y_0\,(\text{say})$$

where Y_0 denotes the initial yield stress. The angular speed necessary for inner-plastic-zone is given by:

$$\bar{p}=\left|\frac{\Omega^2}{3}(x_1^3-1)\left[\frac{1}{3}\frac{(1-C)(5-4C)}{(3-2C)}-1\right]+\frac{Q_1(x_1^{s+3}-1)(1-C)}{(s+2)(s+3)}\times\right.$$
$$\times\left\{\frac{(1-C)^2(s+4)}{(s+2)(3-2C)}-\frac{(2-C)^2}{(s+3)}\right\}-\frac{Q_1(2-C)(x_1^2-1)}{2}\times$$
$$\left.\times\left\{\frac{1}{2}+\frac{x_1(1-C)^3}{(3-2C)}\right\}-\frac{Q_1(2-C)}{(s+2)(s+3)}\left\{1-\frac{s+3}{s+2}+\frac{1}{(s+2)}x_1^{s+2}\right\}-x_1\right|$$

where $Q_1 = E\alpha q_0a^2/kY_0$. Stresses and radial displacement for inner plastic-zone are obtained in non-dimensional form as:

$$\sigma_r=\frac{\Omega^2_{\text{in.pl.zone}}(x_1^3-1)(1-C)(5-4C)}{9x_1(3-2C)}-\frac{\Omega^2_{\text{in.pl.zone}}(x_1^3-1)}{3x_1}-\frac{\bar{p}}{x_1}+$$
$$+\frac{Q_1(1-C)(x_1^{s+3}-1)}{(s+2)(s+3)x_1}\left\{\frac{(1-C)^2(s+4)}{(s+2)(3-2C)}-\frac{(2-C)^2}{(s+3)}\right\}-\frac{Q_1(2-C)(x_1^2-1)}{2x_1}\times$$
$$\times\left[\frac{1}{2}+x_1\frac{(1-C)^3}{(3-2C)}\right]-\frac{Q_1(2-C)}{(s+2)(s+3)x_1}\left[1-\frac{(s+3)}{(s+4)}+\frac{1}{(s+2)}x_1^{s+2}\right]$$

$$\sigma_\theta=\frac{\Omega^2_{\text{in.pl.zone}}(x_1^3-1)(1-C)^2(5-4C)}{9x_1(3-2C)(2-C)}-\frac{\Omega^2_{\text{in.pl.zone}}(1-C)}{3x_1(2-C)}-$$
$$-\frac{Q_1(1-C)(2-C)(x_1^{s+3}-1)}{(s+2)(s+3)^2(3-2C)x_1}+\frac{Q_1(x_1^2-1)}{2}\left\{\frac{2-C}{2}-\frac{(1-C)^2}{(2-C)(3-2C)}\right\}+$$
$$+Q_1(x_1^2-1)\left\{\frac{(1-C)^4(s+4)}{(3-2C)(s+2)^2(s+3)(2-C)}-\frac{(2-C)}{(s+2)^2}\right\}+\frac{Q_1(1-C)}{x_1(s+2)(s+3)}-$$
$$-\frac{(1-C)}{x_1(2-C)}\bar{p}$$

$$\bar{u}=\frac{(1-C)}{(2-C)}\left[\frac{\Omega^2_{\text{in.pl.zone}}}{3}x_1-\frac{Q_1(1-C)x_1}{2}\times\right.$$
$$\left.\times\left\{-1+x_1^2-\frac{2(s+4)}{(s+2)^2(s+3)}x_1^{s+2}\right\}\right] \tag{25}$$

where $\sigma_r=\dfrac{T_{rr}}{Y_0}$, $\sigma_\theta=\dfrac{T_{\theta\theta}}{Y_0}$, $u=\dfrac{\bar{u}Y_0a}{E}$, $\bar{p}=\dfrac{p}{Y_0}$.

Intermediate plastic zone, T_{zz} (=0), $T_{rr} = T_{\theta\theta}$; $r_1 < r \le r_2$: The onset of yielding occurs between radiuses r_1 and r_2, and the compressive radial stress T_{rr} is equal to the compressive circumferential stress $T_{\theta\theta}$. Accordingly, these stresses are equal to the yield stress. Therefore, the radial and circumferential stress are neither maximum nor minimum in that zone. The stresses and radial displacement for intermediate-zone in non-dimensional form are given as:

$$\sigma_r = \frac{\Omega^2_{\text{int.med.pl.zone}}(x_2^3-1)(1-C)(5-4C)}{9x_2(3-2C)} - \frac{\Omega^2_{\text{int.med.pl.zone}}(x_2^3-1)}{3x_2} - \frac{\bar{p}}{x_2} + \frac{Q_1(1-C)(x_2^{s+3}-1)}{(s+2)(s+3)x_2}\left\{\frac{(1-C)^2(s+4)}{(s+2)(3-2C)} - \frac{(2-C)^2}{(s+3)}\right\} - \frac{Q_1(2-C)(x_2^2-1)}{2x_2} \times \left[\frac{1}{2} + x_2\frac{(1-C)^3}{(3-2C)}\right] - \frac{Q_1(2-C)}{(s+2)(s+3)x_2}\left[1 - \frac{(s+3)}{(s+4)} + \frac{1}{(s+2)}x_2^{s+2}\right]$$

$$\sigma_\theta = \frac{\Omega^2_{\text{int.med.pl.zone}}(x_2^3-1)(1-C)^2(5-4C)}{9x_2(3-2C)(2-C)} - \frac{\Omega^2_{\text{int.med.pl.zone}}(1-C)}{3x_2(2-C)} - \frac{Q_1(1-C)(2-C)(x_2^{s+3}-1)}{(s+2)(s+3)^2(3-2C)x_2} + \frac{Q_1(x_2^2-1)}{2}\left\{\frac{2-C}{2} - \frac{(1-C)^2}{(2-C)(3-2C)}\right\} + Q_1(x_2^2-1)\left\{\frac{(1-C)^4(s+4)}{(3-2C)(s+2)^2(s+3)(2-C)} - \frac{(2-C)}{(s+2)^2}\right\} + \frac{Q_1(1-C)}{x_2(s+2)(s+3)} - \frac{(1-C)}{x_2(2-C)}\bar{p}$$

$$\bar{u} = \frac{(1-C)}{(2-C)}\left[\frac{\Omega^2_{\text{int.med.pl.zone}}}{3}x_2 - \frac{Q_1(1-C)x_2}{2} \times \left\{-1 + x_2^2 - \frac{2(s+4)}{(s+2)^2(s+3)}x_2^{s+2}\right\}\right] \qquad (26)$$

where $x_2 = (r_1 < r < r_2)/a$.

Outer plastic zone, T_{zz} (= 0) $< T_{rr} < T_{\theta\theta}$; $r_2 < r < a$. Yielding occurs between radius r_2 and a, and the compressive circumferential stress $T_{\theta\theta}$ is greater than compressive radial stress T_{rr}. It has been seen that $|T_{\theta\theta}|$ has a maximum value at the $r < a$ and $r > r_2$. For yielding at $r_2 < r < a$, Eq.(24) becomes, as stresses and radial displacement for outer plastic-zone are obtained in non-dimension form as:

$$\sigma_r = \frac{\Omega^2_{\text{out.pl.zone}}(x_3^3-1)(1-C)(5-4C)}{9x_3(3-2C)} - \frac{\Omega^2_{\text{out.pl.zone}}(x_3^3-1)}{3x_3} - \frac{\bar{p}}{x_3} + \frac{Q_1(1-C)(x_3^{s+3}-1)}{(s+2)(s+3)x_3}\left\{\frac{(1-C)^2(s+4)}{(s+2)(3-2C)} - \frac{(2-C)^2}{(s+3)}\right\} - \frac{Q_1(2-C)(x_3^2-1)}{2x_3} \times \left[\frac{1}{2} + x_3\frac{(1-C)^3}{(3-2C)}\right] - \frac{Q_1(2-C)}{(s+2)(s+3)x_3}\left[1 - \frac{(s+3)}{(s+4)} + \frac{1}{(s+2)}x_3^{s+2}\right]$$

$$\sigma_\theta = \frac{\Omega^2_{\text{out.pl.zone}}(x_3^3-1)(1-C)^2(5-4C)}{9x_3(3-2C)(2-C)} - \frac{\Omega^2_{\text{out.pl.zone}}(1-C)}{3x_3(2-C)} - \frac{Q_1(1-C)(2-C)(x_3^{s+3}-1)}{(s+2)(s+3)^2(3-2C)x_3} + \frac{Q_1(x_3^2-1)}{2}\left\{\frac{2-C}{2} - \frac{(1-C)^2}{(2-C)(3-2C)}\right\} + Q_1(x_3^2-1)\left\{\frac{(1-C)^4(s+4)}{(3-2C)(s+2)^2(s+3)(2-C)} - \frac{(2-C)}{(s+2)^2}\right\} + \frac{Q_1(1-C)}{x_3(s+2)(s+3)} - \frac{(1-C)}{x_3(2-C)}\bar{p}$$

$$\bar{u} = \frac{(1-C)}{(2-C)}\left[\frac{\Omega^2_{\text{out.pl.zone}}}{3}x_3 - \frac{Q_1(1-C)x_3}{2} \times \left\{-1 + x_3^2 - \frac{2(s+4)}{(s+2)^2(s+3)}x_3^{s+2}\right\}\right] \qquad (27)$$

where $x_3 = (r_2 < r < a)/a$.

NUMERICAL RESULTS AND DISCUSSION

The results are presented in terms of the following dimensionless and normalized variables: radial coordinate $R = r/b$, $x_1 = r_1/a$; $x_2 = r_2/a$ and $x_3 = (r_2 < r < a)/a$, angular velocity $\Omega^2 = \rho\omega^2 a^2/Y_0$, stress $\sigma_r = T_{rr}/Y_0$, $\sigma_\theta = T_{\theta\theta}/Y_0$; displacement $u = u$□$Y_0 a/E$ pressure p□ $= p/Y_0$, $Q_1 = E\alpha q_0 a^2/kY_0$ and $s = 2$. In numerical calculations for a solid disk with heat source subjected to pressure, the following values are calculated (Table 1):

- for $C = 0$ (incompressible material), $s = 2$: when $Q_1 = 1$ then $\Omega^2 = 1.34$ and p□ = 0.96; when $Q_1 = 8$ then $\Omega^2 = 4.47$ and p□ = 0.25; when $Q_1 = 10$ then $\Omega^2 = 1.3$ and p□ = 0.142.
- for $C = 0.25$ (compressible material), $s = 2$: when $Q_1 = 1$ then $\Omega^2 = 1.3$ and p□ = 0.876; when $Q_1 = 8$ then $\Omega^2 = 4.04$ and p□ = 0.379; when $Q_1 = 10$ then $\Omega^2 = 4.03$ and p□ = 0.20.
- for $C = 0.5$ (compressible material), s =2: when $Q_1 = 1$ then $\Omega^2 = 1.25$ and p□ = 0.893; when $Q_1 = 8$ then $\Omega^2 = 3.54$ and p□ = 0.311; when $Q_1 = 10$ then $\Omega^2 = 3.5$ and p□ = 0.28.

Curves are drawn in Fig. 2, depicting angular speed vs. radii ratio $R = r/a$. It has been seen that incompressible material requires higher angular speed as compared to the incompressible material. With the effect of heat generation, the values of angular speed must be increased at the external surface of the solid disc. From Fig. 3, curves are drawn depicting the pressure required on the solid disk vs. radii ratio $R = r/b$. With the effect of heat, the solid disk requires high pressure at the outer plastic surface. In Fig. 4, curves are drawn for stresses and displacement with respect to radii ratio $R = r/b$ for elastic-plastic transition and fully plastic state, respectively. It has been seen that the displacement maximum at the outer surface for $Q_1 = 1$ and radial stress maximum at the inner plastic surface for compressible material as compared to incompressible material for $Q_1 = 8.10$. Heat increases the values of stresses at the inner surface of the solid disk. The solid disk is likely to fracture at the origin.

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Table 1. Numerical calculated values for parameters of solid disk with heat source subjected to pressure.

$R = r/a$			$Q_1 = 1$		$Q_1 = 8$		$Q_1 = 10$	
		Compressibility C	Angular speed required	Pressure required p□	Angular speed required	Pressure required p□	Angular speed required	Pressure required p□
	0	0	1.34	0.955	4.47	0.936	4.5	0.932
	0.1	0	1.34	0.953	4.47	0.937	4.5	0.934
	0.2	0	1.34	0.949	4.47	0.9382	4.5	0.9348
	0.3	0	1.34	0.944	4.47	0.9387	4.5	0.935
	0.4	0	1.34	0.938	4.47	0.941	4.5	0.937
	0.47	0	1.34	0.933	4.47	0.944	4.5	0.942
	0.48839	0	1.34	0.933	4.47	0.944	4.5	0.942
	0.4884	0	1.34	0.933	4.47	0.944	4.5	0.942
	0.488841	0	1.34	0.933	4.47	0.944	4.5	0.942
	0.488842	0	1.34	0.933	4.47	0.944	4.5	0.942
	0.488843	0	1.34	0.933	4.47	0.944	4.5	0.942
	0.488844	0	1.34	0.933	4.47	0.944	4.5	0.942
	0.5	0	1.34	0.933	4.47	0.944	4.5	0.942
	0.6	0	1.34	0.929’	4.47	0.959	4.5	0.957
	0.7	0	1.34	0.975	4.47	0.995	4.5	0.995
	0.8	0	1.34	0.866	4.47	1.1	4.5	1.11
	0.9	0	1.34	0.869	4.47	2.12	4.5	1.99
	1	0	1.34	0.96	4.47	0.25	4.5	0.142
	0	0.25	1.3	0.965	4.04	0.948	4.03	0.93857
	0.1	0.25	1.3	0.962	4.04	0.947	4.03	0.93859
	0.2	0.25	1.3	0.958	4.04	0.946	4.03	0.9379
	0.3	0.25	1.3	0.954	4.04	0.945	4.03	0.9373
	0.4	0.25	1.3	0.949	4.04	0.944	4.03	0.938
	0.47	0.25	1.3	0.945	4.04	0.947	4.03	0.941
	0.48839	0.25	1.3	0.945	4.04	0.947	4.03	0.941
	0.4884	0.25	1.3	0.945	4.04	0.947	4.03	0.941
	0.488841	0.25	1.3	0.945	4.04	0.947	4.03	0.941
	0.488842	0.25	1.3	0.945	4.04	0.947	4.03	0.941
	0.488843	0.25	1.3	0.945	4.04	0.947	4.03	0.941
	0.488844	0.25	1.3	0.945	4.04	0.947	4.03	0.941
	0.5	0.25	1.3	0.945	4.04	0.947	4.03	0.941
	0.6	0.25	1.3	0.942	4.04	0.96	4.03	0.956
	0.7	0.25	1.3	0.982	4.04	0.955	4.03	0.995
	0.8	0.25	1.3	0.876	4.04	1.11	4.03	1.120
	0.9	0.25	1.3	0.883	4.04	3.1	4.03	2.34
	1	0.25	1.3	0.876	4.04	0.379	4.03	0.20
	0	0.5	1.25	0.980	3.54	0.944	3.5	0.944
	0.1	0.5	1.25	0.971	3.54	0.9433	3.5	0.943
	0.2	0.5	1.25	0.969	3.54	0.9433	3.5	0.942
	0.3	0.5	1.25	0.966	3.54	0.942	3.5	0.941
	0.4	0.5	1.25	0.963	3.54	0.941	3.5	0.939
	0.47	0.5	1.25	0.956	3.54	0.945	3.5	0.942
	0.48839	0.5	1.25	0.956	3.54	0.945	3.5	0.942
	0.4884	0.5	1.25	0.956	3.54	0.945	3.5	0.942
	0.488841	0.5	1.25	0.956	3.54	0.945	3.5	0.942
	0.488842	0.5	1.25	0.956	3.54	0.945	3.5	0.942
	0.488843	0.5	1.25	0.956	3.54	0.945	3.5	0.942
	0.488844	0.5	1.25	0.956	3.54	0.945	3.5	0.942
	0.5	0.5	1.25	0.956	3.54	0.945	3.5	0.942
	0.6	0.5	1.25	0.955	3.54	0.959	3.5	0.956
	0.7	0.5	1.25	0.988	3.54	0.995	3.5	0.995
	0.8	0.5	1.25	0.881	3.54	1.11	3.5	1.126
	0.9	0.5	1.25	0.897	3.54	2.52	3.5	2.865
	1	0.56	1.25	0.893	3.54	0.311	3.5	0.28

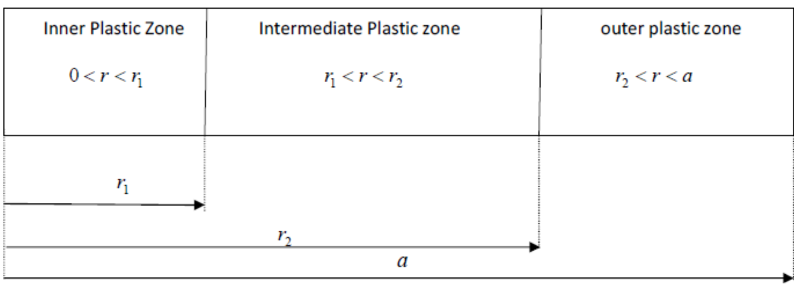

Figure 1. Scheme of general elastic-plastic case in the solid disk: inner plastic zone, intermediate plastic zone, and outer plastic zone.

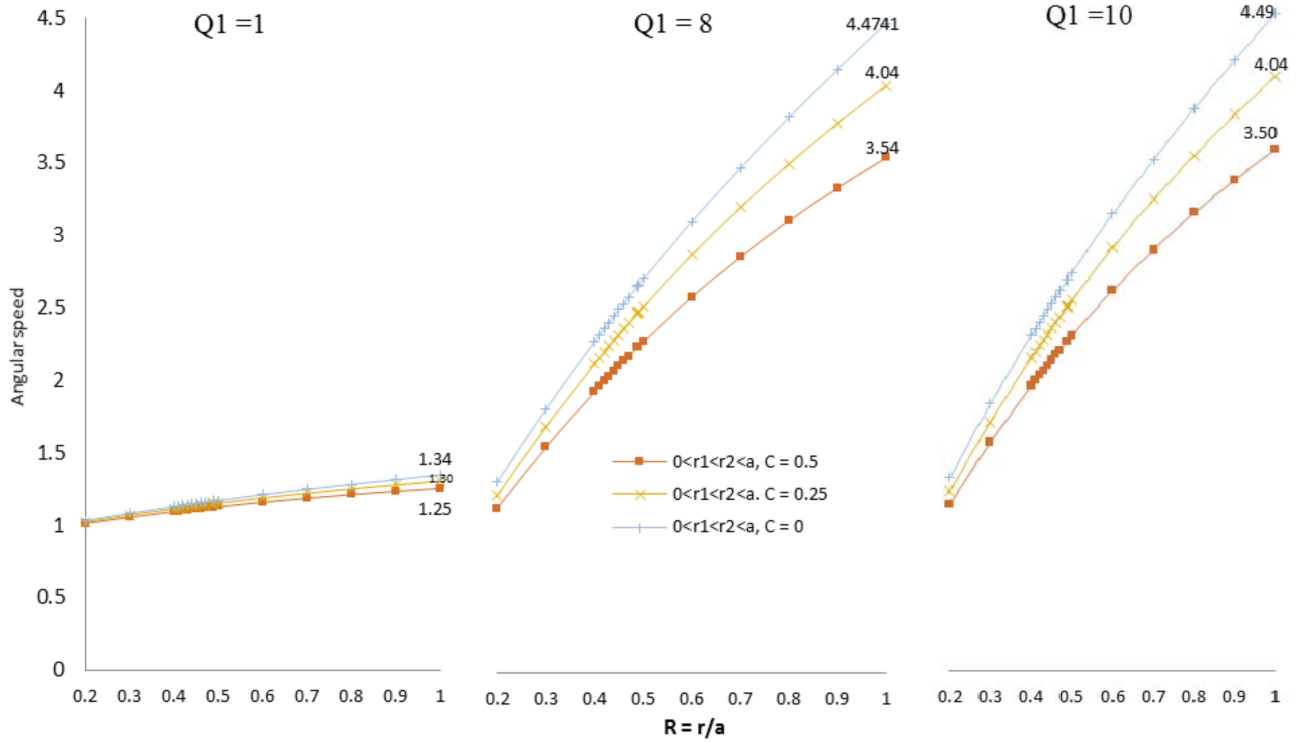

Figure 2. Angular speed required for the solid disk.

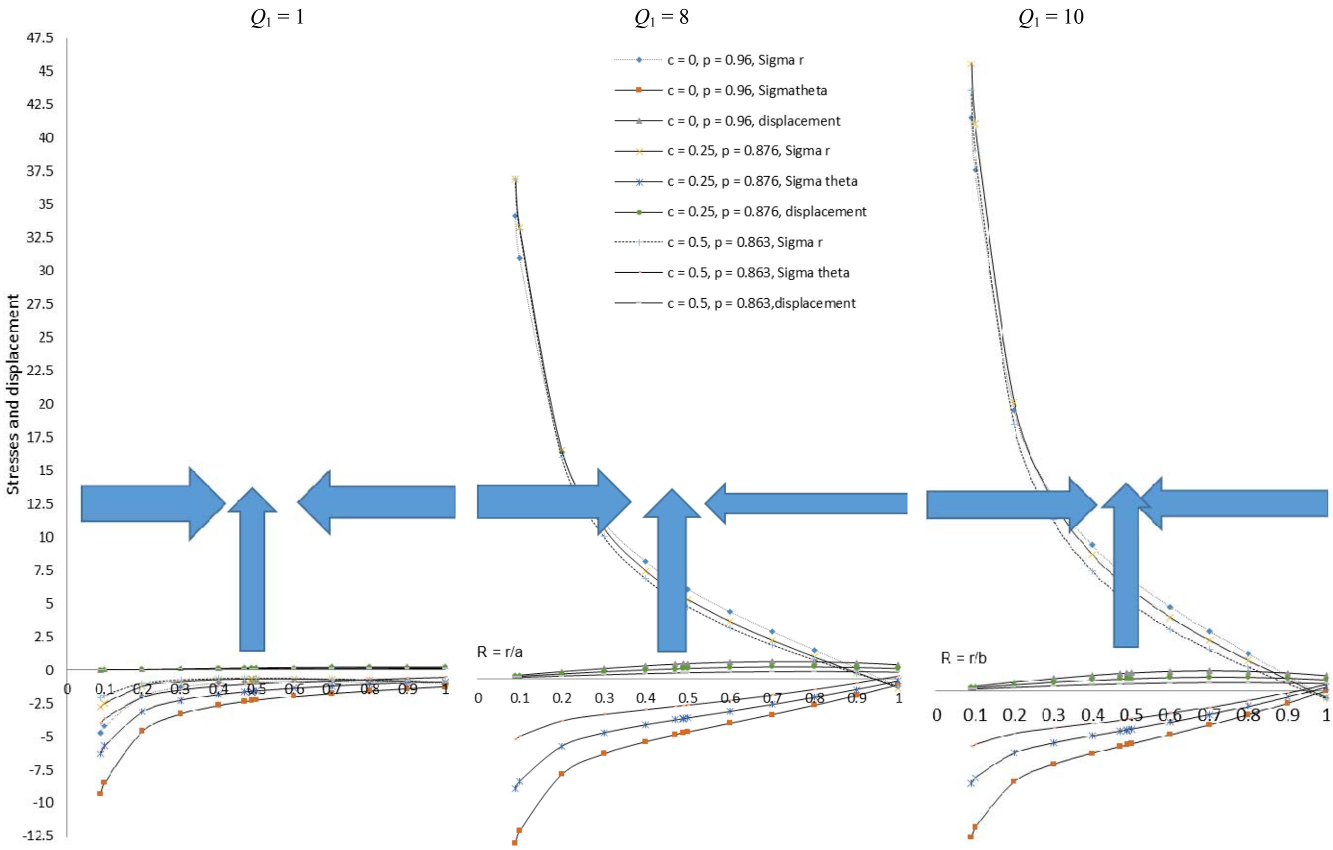

Figure 3. Stresses and displacement distribution in a solid disk along the radii ratio $R = r/b$.

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