Thakur Pankaj[1], S.B. Singh[2], Jatinder Kaur[3]

ELASTIC-PLASTIC TRANSITIONAL STRESSES IN A THIN ROTATING DISC WITH SHAFT HAVING VARIABLE THICKNESS UNDER STEADY STATE TEMPERATURE

ELASTOPLASTIČNI PRELAZNI NAPONI U TANKOM ROTIRAJUĆEM DISKU SA VRATILOM SA PROMENLJIVOM DEBLJINOM POD RAVNOMERNIM UTICAJEM TEMPERATURE

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Adresa autora / Author's address:
[1)] Department of Mathematics, IEC University Badii, Solan, Himachal Pradesh, PIN: 174103, India
dr_pankajthakur@yahoo.com
[2)] Department of Mathematics, Punjabi University Patiala, Punjab -147002, India

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Abstract

Stresses for the elastic-plastic transition and fully plastic state have been derived for a thin rotating disc having variable thickness with shaft at different temperatures. Results have been discussed and depicted graphically. It has been observed that in the absence of thickness, the rotating disc made of incompressible material e.g. rubber with inclusion require a higher angular speed to yield at the internal surface as compared to a disc of compressible material e.g. copper, brass and steel and a much higher angular speed is required to yield with the increase in radii ratio. With the effect of variation thickness, reverse results and lesser angular speed are required to yield at the internal surface. With the introduction of thermal effects for k = 0, lesser angular speed is required to yield at the internal surface. When thickness varies (say k = 2.5, 5), reverse in the results. The radial stresses is maximum at the internal surface with no thickness, when thickness varies the circumferential stress is maximum at the internal surface.

Izvod

Elasto-plastični prelazni naponi i stanje potpune plastičnosti određeni su za tanki rotirajući disk promenljive debljine sa vratilom na različitim temperaturama. Rezultati su diskutovani i prikazani grafički. Uočava se da je, u odsustvu debljine, rotirajući disk od nestišljivog materijala, na pr. guma sa uključcima, potrebna veća ugaona brzina za tečenje na unutrašnjoj površini u poređenju sa diskom od stišljivog materijala, na pr. bakra, mesinga ili čelika, a sa druge strane, potrebna je mnogo veća ugaona brzina za tečenje sa porastom odnosa poluprečnika. Sa promenljivom debljinom, dobijaju se suprotni rezultati kao i manja ugaona brzina za pojavu tečenja na unutrašnjoj površini. Uvođenjem termičkih uticaja pri k = 0, potrebna je manja ugaona brzina za tečenje na unutrašnjoj površini. Kada se debljina menja (k = 2,5 i 5) rezultati su suprotni. Radijalne komponente napona su maksimalne na unutrašnjoj površini bez debljine, a sa promenom debljine, napon u obimnom pravcu je maksimalan na unutrašnjoj površini.

INTRODUCTION

Rotating discs form an essential part of the design of rotating machinery, namely rotors, turbines, compressors, flywheel and computer disc drive etc. The analysis of thin rotating discs made of isotropic material has been discussed extensively by Timoshenko and Goodier /1/ in the elastic range and by Chakrabarty /2/ and Heyman /3/ for the plastic range. Their solution for the problem of fully plastic state does not involve the plane stress condition, that is to say, we can obtain the same stresses and angular velocity required by the disc to become fully plastic without using the plane stress condition (i.e. $T_{xx} = 0$). Gupta and Shukla /4/ obtained a different solution for the fully plastic state by using Seth's transition theory and plane stress condition. This theory, /5/, does not require any assumptions as a yield condition, incompressibility condition, and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. It utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems /4, 5, 10-31/.

Seth, /6/, has defined the generalized principal strain measure as:

$$e_{ii} = \int_0^{e_{ii}^A} \left[1-2e_{ii}^A\right]^{\frac{n}{2}-1} de_{ii}^A = \frac{1}{n}\left[1-\left(1-2e_{ii}^A\right)^{\frac{1}{n}}\right], \ (i=1,2,3) \quad (1)$$

where 'n' is the measure, and e_{ij}^A are Almansi finite strain components. For $n = -2, -1, 0, 1, 2$ it gives Cauchy, Green Hencky, Swainger and Almansi measures, respectively.

Here, elastic-plastic transition stresses in a thin rotating disc of variable thickness with shaft, under steady state temperature are investigated by Seth's transition theory. The thickness of disc is assumed to vary along the radius in the form:

$$h = h_0 (r/b)^{-k} \tag{2}$$

Where h_0 is the thickness at $r = b$ and k is the thickness parameter. Results are numerically obtained and depicted graphically.

MATHEMATICAL MODEL

We consider a thin disc of variable thickness with central bore of radius a and external radius b. The annular disc is mounted on a rigid shaft. The disc is rotating with angular speed ω of gradually increasing magnitude about an axis perpendicular to its plane and passing through the centre as shown in Fig. 1. The disc is so thin that it is effectively in a state of plane stress ($T_{zz} = 0$) and variation of thickness is radial and symmetric with respect to the mid plane. The temperature applied for central bore of the disc is θ.

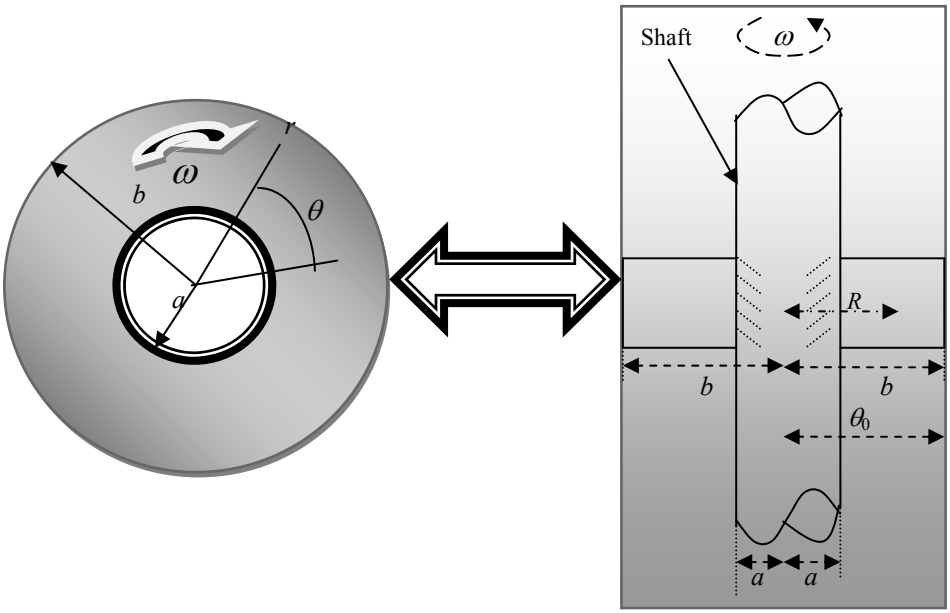

Figure 1. Geometry and scheme of rotating disc with concentric circular hole.
Slika 1. Geometrija i shema rotirajućeg diska sa koncentričnim otvorom

BOUNDARY CONDITIONS

The disk considered in the present study is of variable thickness and is subjected to a temperature gradient field. The inner surface of the disk is assumed to be fixed to a shaft. The outer surface of the disk is free from mechanical load. Thus, the boundary conditions of the problem are given by:

$$\text{(i)}\ \ r = a \cdot u = 0$$
$$\text{(ii)}\ \ r = b,\ \ T_{rr} = 0 \tag{3}$$

where u and T_{rr} denote displacement and stress along the radial direction.

FORMULATION OF THE PROBLEM

The displacement components in cylindrical polar coordinate are given by Seth, /6/:

$$u = r(1-\beta),\ v = 0,\ w = dz \tag{4}$$

where β is function of $r = \sqrt{x^2 + y^2}$ only and d is a constant.

The finite strain components are given by Seth /6/ as:

$$e_{rr}^A \equiv \frac{\partial u}{\partial r} - \frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2 = \frac{1}{2}\left[1-\left(r\beta' + \beta\right)^2\right],$$

$$e_{\theta\theta}^A \equiv \frac{u}{r} - \frac{u^2}{2r^2} = \frac{1}{2}\left[1-\beta^2\right],$$

$$e_{zz}^A \equiv \frac{\partial w}{\partial z} - \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \frac{1}{2}\left[1-\left(1-d\right)^2\right],$$

$$e_{r\theta}^A = e_{\theta z}^A = e_{zr}^A = 0 \tag{5}$$

where $\beta' = d\beta/dr$ and the meaning of the superscript *"A"* is for Almansi.

Substituting Eqs.(5) in Eq.(1), the generalized components of strain are:

$$e_{rr} = \frac{1}{n}\left[1-\left(r\beta' + \beta\right)^n\right],$$

$$e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right],$$

$$e_{zz} = \frac{1}{n}\left[1-(1-d)^n\right],$$

$$e_{r\theta} = e_{\theta z} = e_{zr} = 0 \tag{6}$$

The stress-strain relations for isotropic material are given by /32/,

$$T_{ij} = \lambda\delta_{ij} I_1 + 2\mu e_{ij} - \xi\theta\delta_{ij},\ \ (i,j = 1,2,3) \tag{7}$$

where T_{ij} are the stress components, λ and μ are Lame's constants, $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is Kronecker's delta, $\xi = \alpha(3\lambda + 2\mu)$, α being the coefficient of thermal expansion, and θ is the temperature. Further, θ has to satisfy:

$$\nabla^2\theta=0$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d^2\theta}{dr^2}\equiv\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right)=0$$

or

$$\frac{d\theta}{dr}=\frac{D_1}{r}$$

which has solutions

$$\theta=D_1\left(\log r+D_2\right) \tag{8}$$

where D_1 and D_2 are constants of integration and can be determined from the boundary condition.

Equations (7) for this problem become:

$$T_{rr}=\frac{2\lambda\mu}{\lambda+2\mu}[e_{rr}+e_{\theta\theta}]+2\mu e_{rr}-\frac{2\mu\xi\theta}{(\lambda+2\mu)}$$

$$T_{\theta\theta}=\frac{2\lambda\mu}{\lambda+2\mu}[e_{rr}+e_{\theta\theta}]+2e_{\theta\theta}-\frac{2\mu\xi\theta}{(\lambda+2\mu)}$$

$$T_{r\theta}=T_{\theta z}=T_{zr}=T_{zz}=0. \tag{9}$$

Substituting Eq.(5) in Eq.(6), the strain components in terms of stresses are obtained as /10/:

$$e_{rr}=\frac{\partial u}{\partial r}-\frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2=\frac{1}{2}\left[1-(r\beta'+\beta)^2\right]=$$

$$=\frac{1}{E}\left[T_{rr}-\left(\frac{1-C}{2-C}\right)T_{\theta\theta}\right]+\alpha\theta,$$

$$e_{\theta\theta}=\frac{u}{r}-\frac{u^2}{2r^2}=\frac{1}{2}\left[1-\beta^2\right]=\frac{1}{E}\left[T_{\theta\theta}-\left(\frac{1-C}{2-C}\right)T_{rr}\right]+\alpha\theta,$$

$$e_{zz}=\frac{\partial w}{\partial z}-\frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2=\frac{1}{2}\left[1-(1-d)^2\right]=$$

$$=-\frac{(1-C)}{E(2-C)}[T_{rr}-T_{\theta\theta}]+\alpha\theta,$$

$$e_{r\theta}=e_{\theta z}=e_{zr}=0 \tag{10}$$

where E is the Young's modulus and C is compressibility factor of the material in terms of Lame's constant, given by $E=\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$ and $C=\frac{2\mu}{\lambda+2\mu}$.

Substituting Eq.(6) in Eq.(9), one gets

$$T_{rr}=\frac{2\mu}{n}\left[3-2C-\beta^n\left\{1-C+(2-C)(P+1)^n+\frac{nC\xi\theta}{2\mu\beta^n}\right\}\right],$$

$$T_{\theta\theta}=\frac{2\mu}{n}\left[3-2C-\beta^n\left\{2-C+(1-C)(P+1)^n+\frac{nC\xi\theta}{2\mu\beta^n}\right\}\right],$$

and

$$T_{r\theta}=T_{\theta z}=T_{zr}=T_{zz}=0 \tag{11}$$

where $r\beta'=\beta P$.

Equations of equilibrium are all satisfied except:

$$\frac{d}{dr}(rhT_{rr})-hT_{\theta\theta}+\rho\omega^2r^2h=0 \tag{12}$$

where ρ is the density of the rotating disc material.

The temperature satisfying Laplace Eq.(8) with boundary condition

$$\theta=\theta_0 \text{ at } r=a$$

$$\theta=0 \text{ at } r=b$$

where θ_0 is a constant, given by /27/:

$$D_1=\frac{\theta}{\log(a/b)} \text{ and } D_2=-\log b$$

Substituting D_1 and D_2 from Eq.(8), one gets:

$$\theta=\frac{\theta_0\log(r/b)}{\log(a/b)}.$$

Using Eqs.(11) and (12), one gets a non-linear differential equation for β as:

$$\begin{aligned}&(2-C)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta}=\\&=\frac{rh'}{h}\left[3-2C-\beta^n\left\{1-C+(2-C)(P+1)^n\right\}\right]+\frac{n\rho\omega^2r^2}{2\mu}+\\&+\beta^n\left[1-(P+1)^n-np\left\{1-C+(2-C)(P+1)^n\right\}\right]-\\&-\frac{nC\xi\bar{\theta}_0}{2\mu}\left(\frac{rh'}{h}\ln(r/b)+1\right)\end{aligned} \tag{13}$$

where $\bar{\theta}_0=\frac{\theta_0}{\log(a/b)}$ and $r\beta'=\beta P$ (P is function of β and β is function of r) and $\beta'=d\beta/dr$ (P is function of β and β is function of r only).

SOLUTION THROUGH THE PRINCIPAL STRESSES

For finding the plastic stress, the transition function is taken through the principal stress (see Seth's and Hulsurkar /5-9/, Gupta /11-13, 17/, Pankaj /10, 14-31/) at the transition point $P\to\pm\infty$. The transition function R is defined as:

$$\begin{aligned}R&=\frac{n}{2\mu}[T_{\theta\theta}-C\xi\theta]=\\&=\left[(3-2C)-\beta^n\left\{2-C+(1-C)(P+1)^n\right\}-\frac{nC\xi\theta}{\mu}\right]\end{aligned} \tag{14}$$

Taking the logarithmic differentiation of Eq.(14) with respect to r, one gets:

$$\begin{aligned}&\frac{d(\log R)}{dr}=-\left(\frac{n\beta^nP}{r}\right)\times\\&\times\left[\frac{2-C+(1-C)(P+1)^{n-1}\left\{(P+1)+\beta\frac{dP}{d\beta}\right\}}{(3-2C)-\beta^n\left\{2-C+(1-C)(P+1)^n\right\}-\frac{nC\xi\theta}{\mu}}\right]\end{aligned} \tag{15}$$

Substituting the value $dP/d\beta$ from Eq.(13) in Eq.(15) and taking the asymptotic value $P\to\pm\infty$ and integrating, one gets:

$$R=\frac{A_1r^{\nu-1}}{h} \tag{15}$$

where $\nu=(1-C)/(2-C)$ and A_1 is a constant of integration.

From Eqs.(13) and (15) and using Eq.(2), we have

$$T_{\theta\theta}=\left(\frac{2\mu}{n}\right)\frac{A_1 r^{\nu+k-1}}{h_0}+\frac{C\xi\theta_0\log(r/b)}{\log(a/b)} \tag{16}$$

Substituting Eq.(16) in Eq.(12) using Eq.(2) and integrating, one gets:

$$T_{rr}=\left(\frac{2\mu b^{-k}}{n\nu h_0}\right)A_1 r^{\nu-1}-\frac{\rho\omega^2 r^2}{(3-k)}+ \\ +\frac{B_1 b^{-k}}{r h_0 r^{-k}}+\frac{C\xi\theta_0\log(r/b)}{\log(a/b)}-\frac{C\xi\theta_0}{\log(a/b)} \tag{17}$$

where B_1 is a constant of integration.

Substituting Eqs.(16) and (17) in the second equation of Eq.(10), one gets:

$$\beta=\sqrt{1-\frac{2\nu}{E}\left[\frac{\rho\omega^2 r^2}{3-k}-\frac{B_1 b^{-k}}{h_0 r^{1-k}}+\frac{\alpha E\theta_0(2-C)}{\log(a/b)}+\frac{2(2-C)\alpha E\theta_0\log(r/b)}{\log(a/b)}\right]} \tag{18}$$

where $C\xi = \alpha E(2 - C)$.

Substituting Eq.(18) in Eq.(4), one gets Eq.(19):

$$u=r-r\sqrt{1-\frac{2\nu}{E}\left[\frac{\rho\omega^2 r^2}{3-k}-\frac{B_1 b^{-k}}{h_0 r^{1-k}}+\frac{\alpha E\theta_0(2-C)}{\log(a/b)}+\frac{2(2-C)\alpha E\theta_0\log(r/b)}{\log(a/b)}\right]} \tag{19}$$

where $E=\dfrac{2\mu(3-2C)}{(2-C)}$ is the Young's modulus in term of compressibility factor.

Using boundary condition Eq.(2) in Eqs.(17) and (19), one gets:

$$A_1=\frac{\rho\omega^2 n\nu b^k h_0(1-C)(b^{3-k}-a^{3-k})}{2\mu(3-k)b^{\nu}}+ \\ +\frac{\alpha E\theta_0 n(1-C)(b-a)}{2\mu\log(a/b)b^{1-C/2-C}}-\frac{\alpha E\theta_0 na}{\mu b^{1-C/2-C}} \tag{20}$$

$$B_1=\frac{\rho\omega^2 a^{3-k}h_0}{(3-k)b^{-k}}+\frac{\alpha E\theta_0(2-C)a}{\log(a/b)}+\frac{2(2-C)\alpha E\theta_0 a}{(1-C)} \tag{21}$$

Substituting Eqs.(20) and (21) in Eqs.(16), (17), and (19) respectively, one gets the transitional stresses and displacement as:

$$T_{\theta\theta}=\left(\frac{\rho\omega^2\nu r^k(b^{3-k}-a^{3-k})}{(3-k)r}\left(\frac{r}{b}\right)^{\nu}+\alpha E\theta_0(2-C)\left[\frac{\log(r/b)}{\log(a/b)}-\frac{2a}{(2-C)r}\left(\frac{r}{b}\right)^{1-C/2-C}+\frac{(1-C)(b-a)}{r(2-C)\log(a/b)}\left(\frac{r}{b}\right)^{1-C/2-C}\right]\right) \tag{22}$$

$$T_{rr}=\left(\frac{\rho\omega^2 r^k}{(3-k)r}\left[(b^{3-k}-a^{3-k})\left(\frac{r}{b}\right)^{\nu}-r^{3-k}+a^{3-k}\right]+\frac{\alpha E\theta_0(2-C)}{\log(a/b)}\left[\log(r/b)+\frac{(b-a)}{r}\left(\frac{r}{b}\right)^{1-C/2-C}+\frac{a}{r}-1\right]+\right) \\ \left(+\frac{2\alpha E\theta_0(2-C)}{(1-C)}\left[\frac{a}{r}-\frac{a}{r}\left(\frac{r}{b}\right)^{1-C/2-C}\right]\right) \tag{23}$$

$$u=r-r\sqrt{1-\frac{2\nu}{E}\left\{\left(\frac{\rho\omega^2}{(3-k)r^{1-k}}\left[r^{3-k}-a^{3-k}\right]\right)+\frac{\alpha E\theta_0(2-C)(r-a)}{r\log(a/b)}+\frac{2(2-C)\alpha E\theta_0}{(1-C)}\left[\frac{\log(r/b)}{\log(a/b)}-\frac{a}{r}\right]\right\}} \tag{24}$$

and

$$|T_{rr}-T_{\theta\theta}|=\left|\frac{\rho\omega^2 r^k}{(3-k)r}\left[(1-\nu)(b^{3-k}-a^{3-k})\left(\frac{r}{b}\right)^{\nu}-r^{3-k}+a^{3-k}\right]+ \right. \\ \left. +\alpha E\theta_0\left[2\left(\frac{a}{r}\right)\left(\frac{2-C}{1-C}\right)-\frac{2a}{(1-C)r}\left(\frac{r}{b}\right)^{1-C/2-C}+\frac{(b-a)}{r}\left(\frac{r}{b}\right)^{1-C/2-C}+\frac{(2-C)}{\log(a/b)}\left(\frac{a-r}{r}\right)\right]\right| \tag{25}$$

From Eq.(25) it is seen that $|T_{rr}-T_{\theta\theta}|$ is maximum at the internal surface (at $r = a$), therefore yielding of the disc takes place at the internal surface of the disc and Eq.(25) can be written as:

$$|T_{rr}-T_{\theta\theta}|_{r=a}=\left|\frac{\rho\omega^2(1-\nu)(b^{3-k}-a^{3-k})}{(3-k)a^{1-k}}\left(\frac{a}{b}\right)^{\nu}+\alpha E\theta_0\left[\frac{(b-a)}{a\log(a/b)}\left(\frac{a}{b}\right)^{1-C/2-C}-\frac{2}{(1-C)}\left(\frac{a}{b}\right)^{1-C/2-C}+\frac{2(2-C)}{(1-C)}\right]\right|=Y(\text{say})$$

The angular speed necessary for initial yielding is given by:

$$\Omega_i^2=\frac{\rho\omega_i^2 b^2}{Y}=\left(\left|\frac{(3-k)ab^2}{(1-\nu)a^k(b^{3-k}-a^{3-k})}\left(\frac{b}{a}\right)^{\nu}\right|-\left|\frac{3(2-C)}{(1-a^3/b^3)}\left(\frac{\alpha E\theta_0}{Y}\right)\left[\frac{(1-a/b)}{\log(a/b)}+\frac{2(2-C)}{(1-C)}\left(\frac{a}{b}\right)^{1/(2-C)}-\frac{2}{(1-C)}\left(\frac{a}{b}\right)\right]\right|\right) \tag{26}$$

and

$$\omega_i=\frac{\Omega_i}{b}\sqrt{\frac{Y}{\rho}}\,.$$

The disc becomes fully plastic ($C \to 0$) at the external surface and Eq.(25) becomes:

$$\left|T_{rr}-T_{\theta\theta}\right|_{r=b}=\left|\frac{\rho\omega^2(b^{3-k}-a^{3-k})}{2(3-k)b^{1-k}}+\alpha E\theta_0\left[\frac{(b-a)}{b\log(a/b)}-2\left(\frac{a}{b}\right)\right]\right|=Y^*\text{(say)}$$

The angular speed required for fully plastic state is given by:

$$\Omega_f^2=\frac{\rho\omega_f^2b^2}{Y^*}=\left|\frac{2(3-k)b^{3-k}}{(b^{3-k}-a^{3-k})}\right|-\left|\frac{6}{(1-a^3/b^3)}\left(\frac{\alpha E\theta_0}{Y}\right)\left[\frac{(1-a/b)}{\log(a/b)}-2\left(\frac{a}{b}\right)\right]\right| \tag{27}$$

where $\omega_f=\frac{\Omega_f}{b}\sqrt{\frac{Y}{\rho}}$. $\bar{u}=\frac{u}{b}$, $\theta_1=\frac{\alpha E\theta_0}{Y}$, $\Omega^2=\frac{\rho\omega^2b^2}{Y}$, and $H=\frac{Y}{E}$.

We introduce the following non-dimensional components:

$$R=\frac{r}{b},\quad R_0=\frac{a}{b},\quad \sigma_r=\frac{T_{rr}}{Y},\quad \sigma_\theta=\frac{T_{\theta\theta}}{Y},$$

Elastic-plastic transitional stresses, angular speed and displacement from Eqs.(22), (23), (26) and (24) in non-dimensional form become:

$$\sigma_\theta=\left(\frac{\Omega_i^2\nu(1-R_0^{3-k})R^{\nu+k-1}}{(3-k)}\right)+\theta_1(2-C)\left[\frac{(1-R_0)(1-C)}{(2-C)R\log R_0}R^{1-C/2-C}+\frac{\log R}{\log R_0}-\frac{2R_0}{(2-C)R}R^{1-C/2-C}\right] \tag{28}$$

$$\sigma_r=\left[\frac{\Omega_i^2R^{k-1}}{(3-k)}\left[(1-R_0^{3-k})R^\nu-R^{3-k}+R_0^{3-k}\right]+\frac{\theta_1(2-C)}{\log R_0}\left[\log(R)+\frac{(1-R_0)}{R}(R)^{1-C/2-C}+\frac{R_0}{R}-1\right]+\frac{2\theta_1(2-C)}{(1-C)}\left[\frac{R_0}{R}-\frac{R_0}{R}(R)^{1-C/2-C}\right]\right] \tag{29}$$

$$\Omega_i^2=\frac{\rho\omega_i^2b^2}{Y}=\left|\frac{(3-k)R_0^{1-\nu-k}}{(1-\nu)(1-R_0^{3-k})}\right|-\left|\frac{3\theta_1(2-C)}{(1-R_0^3)}\left[\frac{(1-R_0)}{\log R_0}+\frac{2(2-C)}{(1-C)}R_0^{1/(2-C)}-\frac{2}{(1-C)}R_0\right]\right| \tag{30}$$

$$\bar{u}=R-R\sqrt{1-2\nu H\left\{\frac{\Omega_i^2}{(3-k)}R^{k-1}[\mathrm{R}^{3\text{-k}}-R_0^{3-k}]\right\}+\frac{\theta_1(2-C)(R-R_0)}{R\log(R_0)}+\frac{2(2-C)\theta_1}{(1-C)}\left[\frac{\log(R)}{\log(R_0)}-\frac{R_0}{R}\right]}. \tag{31}$$

Stresses, displacement and angular speed for fully plastic state, are obtained from Eqs.(28), (29), (31) and (27) as:

$$\sigma_\theta=\frac{\Omega_f^2(1-R_0^{3-k})R^k}{2\sqrt{R}(3-k)}+2\theta_1\left[\frac{(1-R_0)}{2R\log R_0}R^{1/2}+\frac{\log R}{\log R_0}-\frac{R_0}{R}R^{\frac{1}{2}}\right] \tag{32}$$

$$\sigma_r=\left(\frac{\Omega_f^2R^{k-1}}{(3-k)}\left[(1-R_0^{3-k})\sqrt{R}-R^{3-k}+R_0^{3-k}\right]+\frac{2\theta_1}{\log R_0}\left\{\log R+\frac{(1-R_0)}{R}R^{1/2}+\frac{R_0}{R}-1\right\}+4\theta_1\left\{\frac{R_0}{R}-\frac{R_0}{R}R^{1/2}\right\}\right) \tag{33}$$

$$\bar{u}=R-R\sqrt{1-H\left\{\frac{\Omega_f^2}{(3-k)}R^{k-1}[\mathrm{R}^{3\text{-k}}-R_0^{3-k}]\right\}+\frac{2\theta_1(R-R_0)}{R\log R_0}+4\theta_1\left[\frac{\log R}{\log R_0}-\frac{R}{R_0}\right]} \tag{34}$$

and

$$\Omega_f^2=\left|\frac{2(3-k)}{(1-R_0^{3-k})}\right|-\left|\frac{6\theta_1}{(1-R_0^3)}\left[\frac{(1-R_0)}{\log(R_0)}-2R_0\right]\right| \tag{35}$$

Particular case:

When there is ($k=0$), the transitional stresses from Eqs.(28) to (31) become:

$$\sigma_\theta=\left(\frac{\Omega_i^2\nu(1-R_0^3)R^{\nu-1}}{3}\right)+\theta_1(2-C)\left[\frac{(1-R_0)(1-C)}{(2-C)R\log R_0}R^{1-C/2-C}+\frac{\log R}{\log R_0}-\frac{2R_0}{(2-C)R}R^{1-C/2-C}\right] \tag{36}$$

$$\sigma_r=\left[\frac{\Omega_i^2}{3R}\left[(1-R_0^3)R^\nu-R^3+R_0^3\right]+\frac{\theta_1(2-C)}{\log R_0}\left[\log(R)+\frac{(1-R_0)}{R}(R)^{1-C/2-C}+\frac{R_0}{R}-1\right]+\frac{2\theta_1(2-C)}{(1-C)}\left[\frac{R_0}{R}-\frac{R_0}{R}(R)^{1-C/2-C}\right]\right] \tag{37}$$

$$\Omega_i^2=\frac{\rho\omega_i^2b^2}{Y}=\left|\frac{3R_0^{1-\nu}}{(1-\nu)(1-R_0^3)}\right|-\left|\frac{3\theta_1(2-C)}{(1-R_0^3)}\left[\frac{(1-R_0)}{\log R_0}+\frac{2(2-C)}{(1-C)}R_0^{1/(2-C)}-\frac{2}{(1-C)}R_0\right]\right| \tag{38}$$

and

$$\bar{u}=R-R\sqrt{\left(1-2\nu H\frac{\Omega_i^2}{3R}[R^3-R_0^3]+\frac{\theta_1(2-C)(R-R_0)}{R\log(R_0)}+\frac{2(2-C)\theta_1}{(1-C)}\left[\frac{\log(R)}{\log(R_0)}-\frac{R_0}{R}\right]\right)} \tag{39}$$

For fully plastic state stresses, displacement and angular velocity from equations (32) to (35) become:

$$\sigma_\theta=\frac{\Omega_f^2(1-R_0^3)R^{1/2}}{6}+2\theta_1\left[\frac{(1-R_0)}{2R\log R_0}R^{1/2}+\frac{\log R}{\log R_0}-\frac{R_0}{R}R^{1/2}\right] \tag{40}$$

$$\sigma_r = \left(\frac{\Omega_f^2}{3R}\left[(1-R_0^3)\sqrt{R} - R^3 + R_0^3\right] + \frac{2\theta_1}{\log R_0}\left\{\log R + \frac{(1-R_0)}{R}R^{1/2} + \frac{R_0}{R} - 1\right\} + 4\theta_1\left\{\frac{R_0}{R} - \frac{R_0}{R}R^{1/2}\right\}\right)$$

$$\sigma_r = \frac{\Omega_f^2}{3R}\left[(1-R_0^3)R^{1/2} - R^3 + R_0^3\right] \quad (41) \qquad \bar{u} = R - R\sqrt{1 - H\frac{\Omega_f^2}{3R}[R^3 - R_0^3] + \frac{2\theta_1(R-R_0)}{R\log R_0} + 4\theta_1\left[\frac{\log R}{\log R_0} - \frac{R}{R_0}\right]} \quad (42)$$

where $$\Omega_f^2 = \left|\frac{6}{(1-R_0^3)}\right| - \left|\frac{6\theta_1}{(1-R_0^3)}\left[\frac{(1-R_0)}{\log(R_0)} - 2R_0\right]\right|. \quad (43)$$

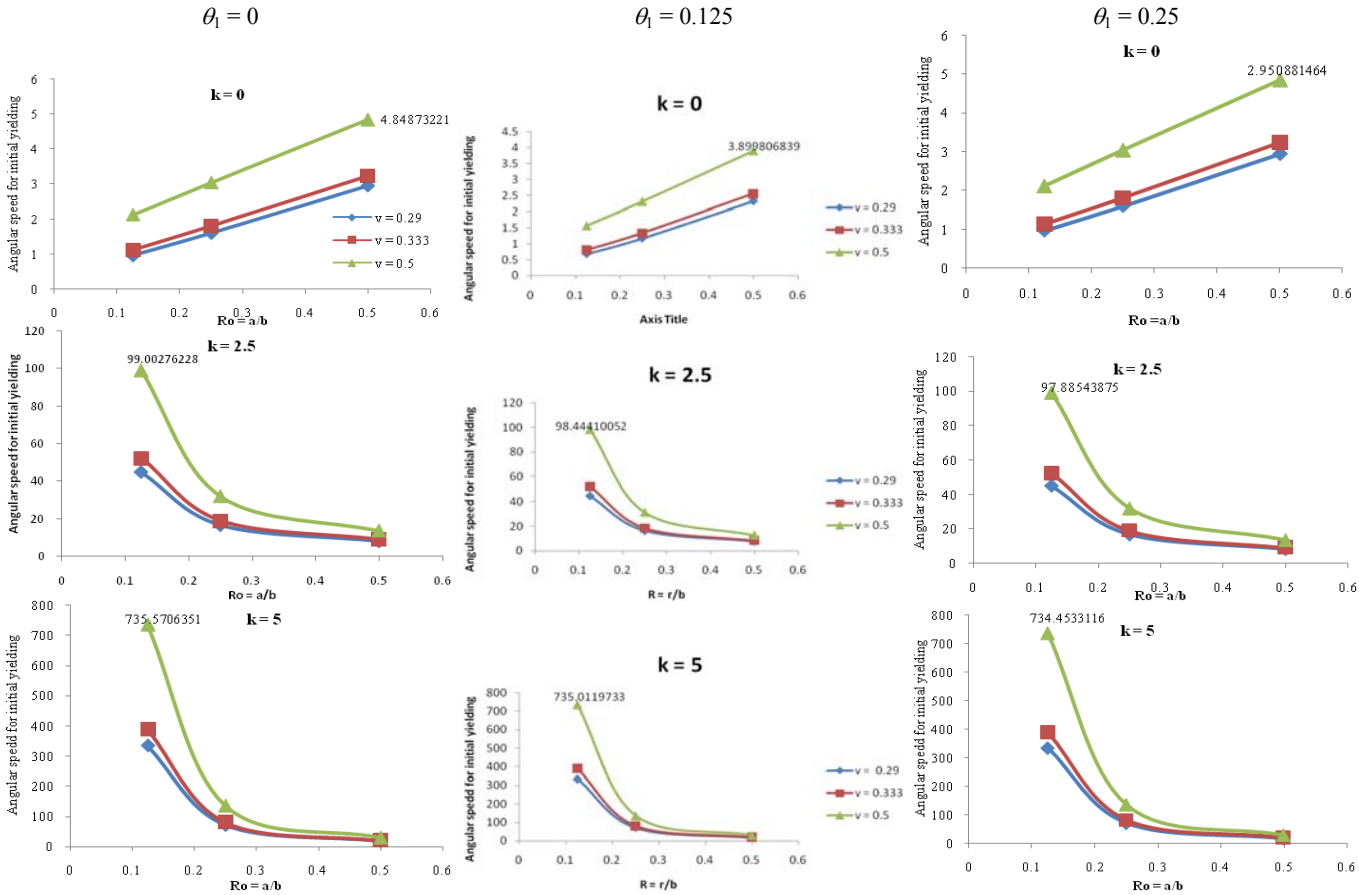

Figure 2. Angular speed required for initial yielding of the rotating disc.
Slika 2. Ugaona brzina potrebna za početak tečenja rotirajućeg diska

Equations (36) to (43) are the same as given by Pankaj /10/.

For calculating the stresses, angular speed based on the above analysis, the following values have been taken as $\nu = 0.5$ (rubber or incompressible material), $\nu = 0.333$ (copper and brass or compressible material), $\nu = 0.29$ (steel or compressible material); $C = 0.00, 0.25, 0.5, 0.75$; $E/Y = 1/2$ and 2; $\theta_0 = 0°F$, 700°F, and $\alpha = 5.0\times10^{-5}$ degF^{-1} (for methyl methacrylate) /33/, $\theta_1 = \alpha E\theta_0/Y = 0, 0.0125, 0.25$ for $E/Y = 1/2$, 2 and $\theta_0 = 0°F$, 700°F, respectively. Curves have been drawn in Fig. 2 between angular speed Ω_i^2 required for initial yielding and various radii ratios $R_0 = a/b$ for $\nu = 0.5$, 0.333, 0.29 and $k = 0$, 2.5, 5 for different values of temperature $\theta_1 = 0$, 0.125, 0.25. It has been observed that in the absence of thickness, the rotating disc made of incompressible material, e.g. rubber, with an inclusion requires higher angular speed to yield at the internal surface as compare to the disc of compressible material, e.g. copper, brass and steel, and a much higher angular speed is required to yield with the increase in radii ratio. With the effect of varied thickness, reverse results and a required lesser angular speed to yield at the internal surface is noted. With the introduction of thermal effects for $k = 0$, lesser angular speed is required to yield at the internal surface. When thickness varies (say $k = 2.5$, 5), results are reversed.

In Figs. 3 and 4, curves have been drawn for stresses and with respect to radii ratio $R = r/b$ for elastic-plastic transition and fully plastic state respectively. It has been seen that radial stresses are maximum at the internal surface with no thickness, when thickness varies the circumferential stress is maximum at the internal surface. With the introduction of thermal effect it decrease the value of radial and circumferential stress at the internal surface and external surface for transitional state, whereas from Fig. 4, it can been seen that the thermal effect increases the values of radial and circumferential stress at the internal surface and external surface for fully-plastic state.

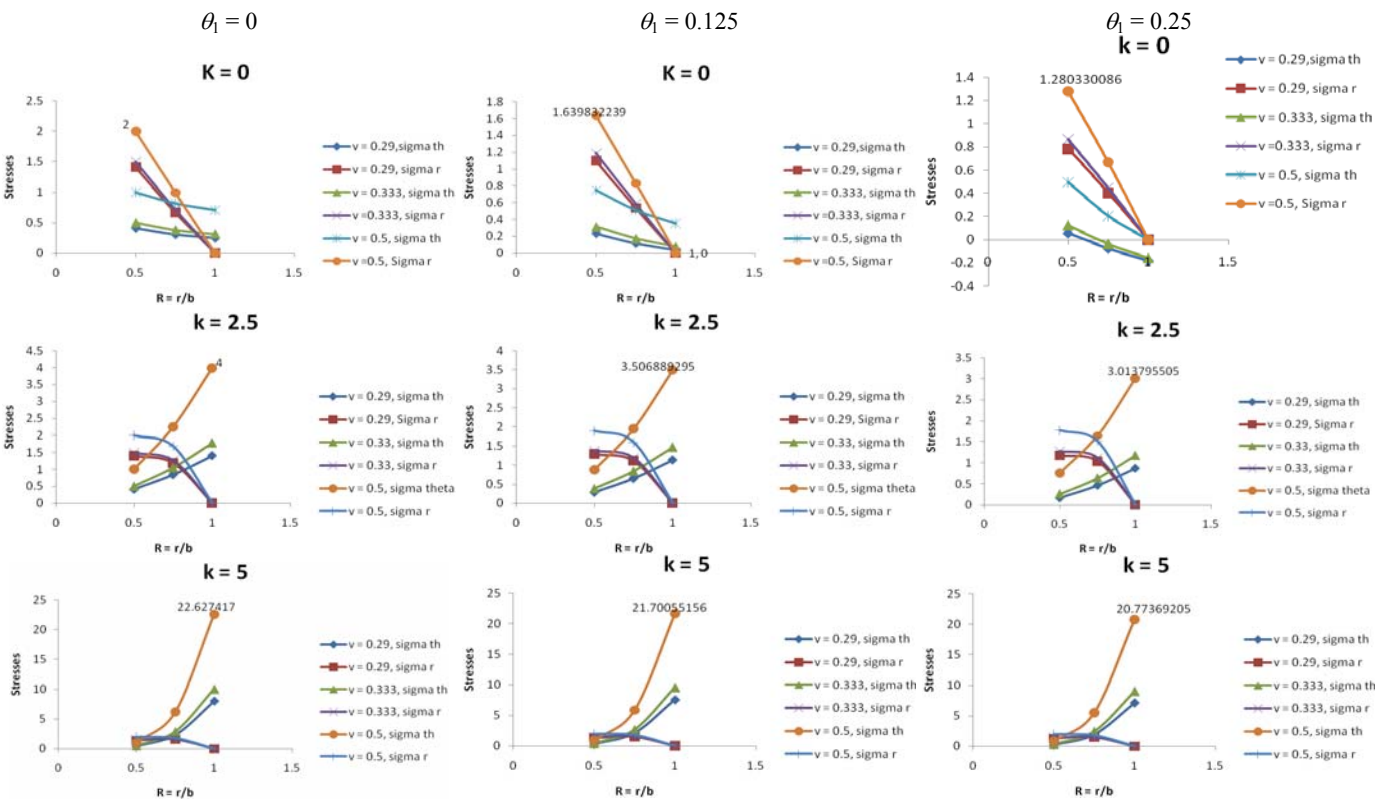

Figure 3. Stresses at the elastic-plastic transition state.
Slika 3. Naponi elastoplastičnog prelaznog stanja

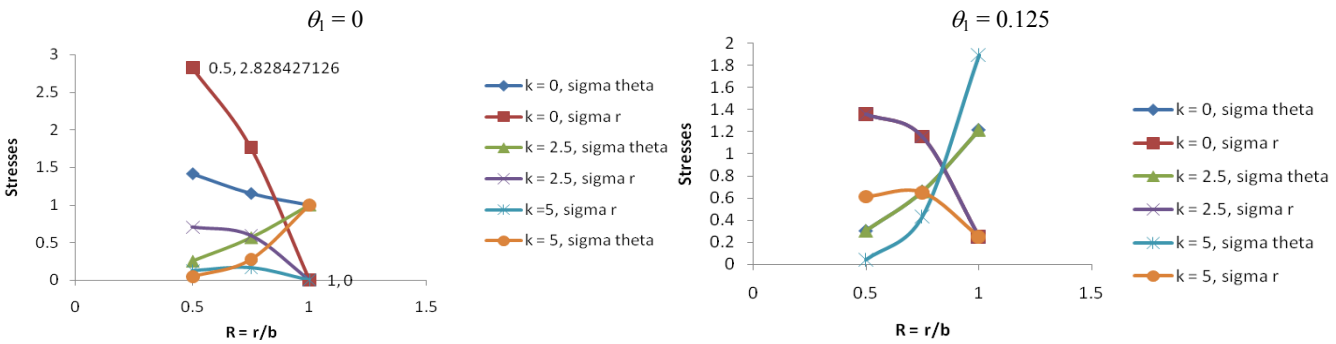

Figure 4. Stresses at the fully plastic state.
Slika 4. Naponi u stanju potpune plastičnosti

CONCLUSION

It has been observed that in the absence of thickness the rotating disc of incompressible material, e.g. rubber, with an inclusion require higher angular speed to yield at the internal surface as compare to disc made of compressible material, e.g. copper, brass and steel and a much higher angular speed is required to yield with the increase in radii ratio. With the effect of thickness variation, results are reversed and a lesser angular speed is required to yield at the internal surface. With the introduction of thermal effects for $k = 0$, lesser angular speed is required to yield at the internal surface. When thickness varies (say $k = 2.5$, 5), reversed are the results. The radial stresses are maximum at the internal surface with no thickness, when thickness varies, the circumferential stress is maximum at the internal surface.

Nomenclature

a, b – internal and external radii of the disc, (m)
u, v, w – displacement components
r, θ, z – radial, circumferential, axial direction
ω – angular velocity of rotation
ρ – density of material
C – compressibility
T_{ij}, e_{ij} – stress and strain rate tensors
Y – yield stress
$\left.\begin{matrix} K_1, K_2 \\ k_1, k_2 \end{matrix}\right\}$ – constants of integration
$\Omega^2 - \rho\omega^2 b^2/E$ (speed factor)
σ_r – radial stress component (T_{rr}/Y)
σ_θ – circumferential stress component ($T_{\theta\theta}/Y$)
θ – temperature
$R = r/b$, $R_0 = a/b$ (radii ratio)

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