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CRACK TIP PLASTIC ZONE UNDER MODE I LOADING AND THE NON-SINGULAR T_{zz}–STRESS

PLASTIČNA ZONA ISPRED VRHA PRSLINE POD I TIPOM OPTEREĆENJA I NESINGULARNI T_{zz}–NAPON

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Ključne reči
- plastična zona
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Abstract

The amplitudes of the second order terms in the three-dimensional expansion of the crack front stress field are the terms T_{xx} and T_{zz} which describe in-plane and out-of-plane constraint, respectively. All previous analyses of the crack tip plastic zone have ignored the effect of T_{zz}-stress. At the same time, the effect of T_{zz}-stress on crack tip plastic zones is not revealed heretofore. It is therefore very important to obtain solutions for crack tip plastic zone size taking into account two components of the T-stresses. The present study focuses on theoretical and numerical analysis of the joint effect of the non-singular T_{xx} and T_{zz}-stress on sizes of the plastic zone in the vicinity of the crack tip under mode I loading conditions. The three-dimensional crack tip stresses including T_{xx} and T_{zz} stresses are incorporated into the von Mises yield criteria to develop an expression that models the crack tip plastic zone. Calculations are performed for three thicknesses of the CT specimen. The predicted sizes of the plastic zone in the vicinity of the crack tip of the analysed CT specimens are bounded by sizes of the plastic zones corresponding to two special conditions, namely, plane stress and plane strain. The theoretical results are compared with the results computed by FEM. Theoretical estimations of the plastic deformation zone size with provision for T-stress components as a whole shows satisfactory results, especially in the line of crack continuation.

Izvod

Amplitude članova drugog reda u trodimenzionalnom razvoju naponskog polja fronta prsline su veličine T_{xx} i T_{zz}, koje opisuju veze u ravni i izvan ravni, respektivno. Sve ranije analize plastične zone ispred vrha prsline su zanemarivale uticaj T_{zz} napona. Tim povodom, ni uticaj T_{zz} napona u plastičnoj zoni ispred vrha prsline nije u potpunosti razjašnjen. Stoga je vrlo bitno pronaći rešenja za dimenzije plastične zone ispred vrha prsline, uzimanjem u obzir ove dve komponente T-napona. Prikazano istraživanje se fokusira na teorijskoj i numeričkoj analizi istovremenog uticaja nesingularnih T_{xx} i T_{zz} napona na veličinu plastične zone u okolini vrha prsline, pod uslovima delovanja opterećenja I tipa. Trodimenzionalni naponi na vrhu prsline, uključujući T_{xx} i T_{zz} napone, su uvršteni u fon Mizesov kriterijum tečenja, radi dobijanja izraza kojim se modelira plastična zona ispred vrha prsline. Proračuni su izvedeni za tri date debljine CT epruvete. Procenjene dimenzije plastične zone u okolini vrha prsline analiziranih CT epruveta su ograničene veličinama plastičnih zona koje se odnose na dva specijalna slučaja, zapravo, ravno stanje napona i ravno stanje deformacije. Teorijski rezultati su poređeni sa rezultatima sračunatim primenom FEM. Teorijske procene dimenzija zone plastične deformacije tretiranjem komponenata T-napona u celini, pokazuju zadovoljavajuće rezultate, posebno u liniji pravca razvoja prsline.

INTRODUCTION

The different sources of a change in in-plane constraint at the crack tip are associated with crack size, geometry of specimen and type of loading. The source of a change of the out-of-plane crack tip constraint is the thickness. To describe in-plane and out-of-plane constraint effects in fracture analysis, the following parameters can be used, namely, T_z-parameter, /1/, local triaxiality parameter h, /2/, and the non-singular terms in William's series expansion of the crack tip stress fields, /3/.

These parameters considerably influence the fracture toughness /4-8/. Not emphasizing the attention to the advantages and disadvantages of the above-mentioned constraint parameters, we concentrate on the non-singular components of the *T*-stresses at the crack tip. The second order terms T_{xx} and T_{zz} in William's series expansion are defined as the *T*-stresses, and they are the only non-zero and non-singular terms. It should be noted that T_{xx} has been simply referred to as *T*-stress. T_{xx} and T_{zz} represent the stresses in the crack surface plane normal to, and tangential to the crack front, respectively.

In a two-dimensional (2D) crack configuration, T_{zz} is related to T_{xx} by $T_{zz} = \nu T_{xx}$ under plane strain conditions, where ν is Poisson's ratio. It is well-known that the sign and magnitude of T_{xx}-stress substantially changes the size and shape of the plane strain crack tip plastic zone /9-11/.

Therefore, the T_{xx}-stress has been used to characterize the effect of in-plane constraint on the crack tip plastic zone.

The amplitudes of the second order terms in the three-dimensional series expansion of the crack front stress field are the terms T_{xx} and T_{zz} which describe in-plane and out-of-plane constraints, respectively. All previous analyses of the crack tip plastic zone have ignored the effect from T_{zz}-stress. At the same time, the effect of the T_{zz}-stress on crack tip plastic zones is not revealed heretofore. It is therefore very important to obtain solutions for crack tip plastic zone size, taking into account two components of the T-stresses.

The present paper focuses on theoretical and numerical analysis of the joint effect of the non-singular T_{xx} and T_{zz}-stresses on sizes of the plastic zone in the vicinity of the crack tip under mode I loading conditions.

THEORETICAL ANALYSIS OF THE PLASTIC ZONE

Modelling of the plastic zone

The general form of the linear elastic crack tip stress fields within a three dimensional crack problem can be characterized by the singular (the first order term) and non-singular terms (second order terms), /3/

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + T_{xx} + \dots, \tag{1}$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + \dots, \tag{2}$$

$$\sigma_{zz} = 2\nu \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} + T_{zz} + \dots, \tag{3}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\frac{3\theta}{2} + \dots, \tag{4}$$

$$\tau_{yz} = 0, \ \tau_{zx} = 0. \tag{5}$$

Here, r and θ are the in-plane polar coordinates of the plane normal to the crack front centred at the crack tip with $\theta = 0$ corresponding to a line ahead of the crack (Fig. 1), σ and τ are the normal and shear stress, respectively, K_I is the mode I stress intensity factor (SIF), E is Young's modulus, ν is Poisson's ratio. The singular term corresponds to the stress intensity factor. The terms T_{xx} and T_{zz} are the amplitudes of the second order terms in the three-dimensional series expansion of the crack front stress field in the x and z directions, respectively. These terms characterize corresponding crack tip constraint along above-mentioned axes.

The T_{xx}-stress component can be calculated from Eqs.(1) and (2) as a difference between σ_{xx} and σ_{yy} stresses very near to the crack front. The value of T_{zz}-stress component is defined according to the following relationship, /4/,

$$T_{zz} = E\varepsilon_{zz} + \nu T_{xx} \tag{6}$$

where ε_{zz} is the strain along the crack front.

In the case of plane stress conditions (2D stress state), the stress component $\sigma_{xx} = 0$. For plane strain conditions, the stress component σ_{zz} is equal to

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = 2\nu \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} + \nu T_{xx} \tag{7}$$

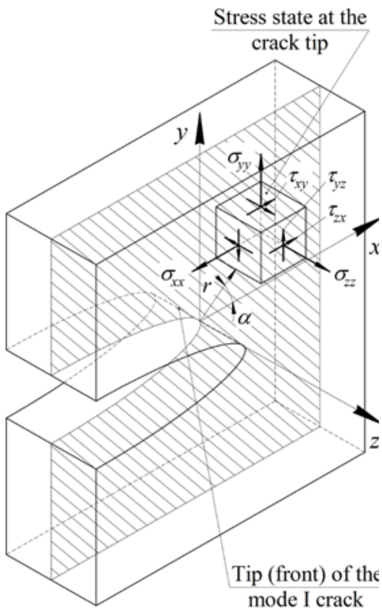


Figure 1. Three-dimensional coordinate system for the region along the crack front.
Slika 1. Trodimenzionalni koordinatni sistem za oblast oko fronta prsline

In the present work, the plastic zone ahead of the crack tip is determined by the von Mises yield criterion

$$\begin{aligned}(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + \\ +6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = 2\sigma_Y^2\end{aligned} \tag{8}$$

where σ_Y is the yield stress.

Substituting Eqs.(1)–(5) into the von Mises yield criterion (8), the crack tip plastic zone size $r = r_p$ can be estimated. After comprehensive mathematical manipulations, the solution for r_p as a function of θ and the T-stress components is given by the following formula

$$\frac{1}{2\pi r_p}\left(K_I^2\left(A_I + \frac{D_I}{K_I}\sqrt{r_p}\right)\right) = 2\sigma_Y^2 \tag{9}$$

The joint effect of the non-singular T_{xx} and T_{zz}-stresses on the sizes of the plastic zone in the vicinity of the crack tip under mode I loading conditions is included into the basic Eq.(9).

The parameters in Eq.(9) are denoted as follows

$$\begin{aligned}D_I = \sqrt{\frac{\pi}{2}}\left(\cos\frac{\theta}{2} + 3\cos\frac{5\theta}{2}\right)T_{xx} - 8\nu\cos\frac{\theta}{2}\left(T_{xx} + \frac{T_{zz}}{\nu}\right) + \\ +16\nu\cos\frac{\theta}{2}(T_{zz})\end{aligned} \tag{10}$$

$$A_I = (1-2\nu)^2(1+\cos\theta) - \frac{3}{4}(\cos 2\theta - 1) \tag{11}$$

Equation (9) can be solved to determine the angular distribution of the plastic zone size in the vicinity of the crack tip

$$r_p(\theta)_{1,2} = \frac{1}{4U^2}\left[V \pm \sqrt{V^2 + 4UW}\right]^2, \tag{12}$$

where parameters U, V, W are

$$U = 4\pi\sigma_Y^2 ,\quad V = K_I D_I ,\quad W = K_I^2 A_I \tag{13}$$

Finally, solution (12) can be written in a more representative form

$$r_p(\theta)_{1,2} = \left[\frac{K_I^2}{\pi\sigma_Y^2}\right]\cdot\left[\frac{D_I \pm \sqrt{D_I^2 + (16\pi\sigma_Y^2)A_I}}{2\sqrt{16\pi\sigma_Y^2}}\right]^2 \tag{14}$$

It can be shown that a special solution for the crack tip plastic zone in the case of plane strain conditions under mode I loading follows from the general solution (12) (e.g., /11/):

$$U = 4\pi\sigma_T^2 ,\quad V = K_I E_I ,\quad W = K_I^2 B_I . \tag{15}$$

Here, the additional coefficients are

$$B_I = (1-2\nu)^2(1+\cos\theta) - \frac{3}{4}(\cos 2\theta - 1) \tag{16}$$

$$E_I = \sqrt{\frac{\pi}{2}}\left(\left(\cos\frac{\theta}{2} + 3\cos\frac{5\theta}{2}\right)T_{xx} - 8\nu\cos\frac{\theta}{2}(2T_{xx}) + 16\nu\cos\frac{\theta}{2}(\nu T_{xx})\right) \tag{17}$$

For plane stress conditions, the angular distribution of the plastic zone at the mode I crack tip can be calculated from (12) taking into account the following coefficients

$$U = 4\pi\sigma_T^2 ,\quad V = K_I F_I ,\quad W = K_I^2 C_I , \tag{18}$$

$$C_I = (1+\cos\theta) - \frac{3}{4}(\cos 2\theta - 1) , \tag{19}$$

$$F_I = \sqrt{\frac{\pi}{2}}\left(\cos\frac{\theta}{2} + 3\cos\frac{5\theta}{2}\right)T_{xx} . \tag{20}$$

It should be noted that Eq.(12) has to meet certain conditions, namely

$$V^2 + 4UW^2 \geq 0 \quad\text{and}\quad U \neq 0 \tag{21}$$

If the above-mentioned inequalities (21) satisfy, the sizes of the crack tip plastic zone are calculated as

$$r_{p1} = \frac{V + \sqrt{V^2 + 4UW}}{2U} ,\quad r_{p2} = \frac{V - \sqrt{V^2 + 4UW}}{2U} . \tag{22}$$

The value of r_{p1} is positive in the wide range of coefficients U, V, W and should be used in all calculations. At the same time the value of r_{p2} has a negative sign.

The effect of thickness on the plastic zone

For the 3D model, the CT specimen is considered. The plastic zone is analysed for a wide range of the ratio B/W = 0.25–0.5 within the limits of pure plane stress and pure plane strain conditions. The stress intensity factor K_{max} of experimental specimens of steel JIS S55C is 66 MPa·m$^{1/2}$, corresponding values of T-stress components are presented in Table 1, /7/. The T-stress components are calculated along the plastic zone boundary on the line of the crack extension. According to the slight variation of T_{xx}-stresses for the CT specimens with different ratio B/W, the value of T_{xx}-stresses has been assumed to be constant and equal to 182 MPa.

The fracture tends to initiate at the specimen thickness centre, the values of the crack tip plastic zone size at specimen thickness centre are chosen to represent the characteristic intensity of these values. The predicted results of the angular distribution of plastic zone sizes in the vicinity of the crack tip at specimen thickness centre, for CT specimens with various thicknesses to width ratio, are given in Fig. 2. To determine the validity of the plastic zone model derived above, the basic equation is applied to a pure mode I situation under plane strain conditions, and the results agree with published solutions, /9, 11/.

Table 1. The T-stress components on the line of the crack extension (boundary of the plastic zone).
Tabela 1. Komponente T-napona u liniji rasta prsline (granica plastične zone)

B/W	0.25	0.40	0.50
T_{xx} (MPa)	186.59	182.36	176.28
T_{zz} (MPa)	–159.47	–106.81	–84.97

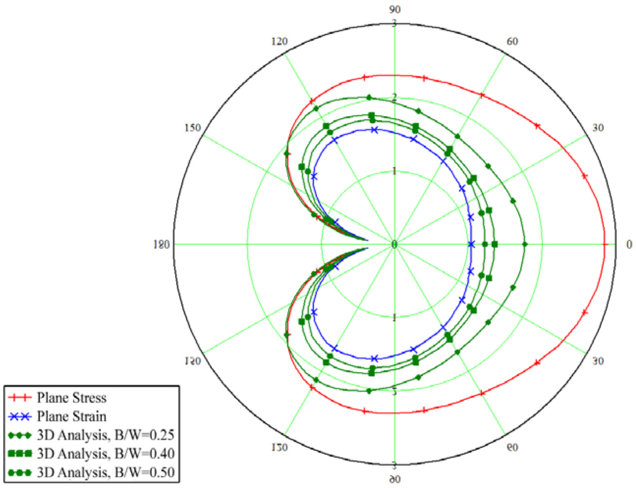


Figure 2. The angular distribution of the plastic zone size at the specimen thickness centre of CT specimens.
Slika 2. Ugaona raspodela veličine plastične zone u sredini debljine CT epruvete

The predicted sizes of the plastic zone in the vicinity of the crack tip of the analysed CT specimens are bounded by sizes of plastic zones corresponding to two special conditions, namely, plane stress and plane strain. Moreover, the shape and size of the plastic zones tend to typical plastic zones for plane strain conditions when specimen thickness increases. Thus, the results confirm the necessity to take into account the constraint effect at the crack tip on the plastic zone by means of both non-singular stresses T_{xx} and T_{zz}.

NUMERICAL MODELLING THE PLASTIC ZONE

Finite Element Analysis

To demonstrate the validity of the plastic zone model, finite element analysis is conducted.

It is well-known that the creation of the calculation model lies in the basis of a numerical experiment. Primary tasks which are being solved for this purpose are the following. First of all, the solid-state geometrical model of the CT specimen (Fig. 3) is created in the modern CAD system. Three geometrical models with the ratio B/W =

0,25; B/W = 0.40 and B/W = 0.50 are prepared for an analysis of the effect of the T-stress components on the plastic zone. Deviation from the figure was that the crack length a, was set at the nominal value of 12.5 mm (a/W = 0.5) for all cases.

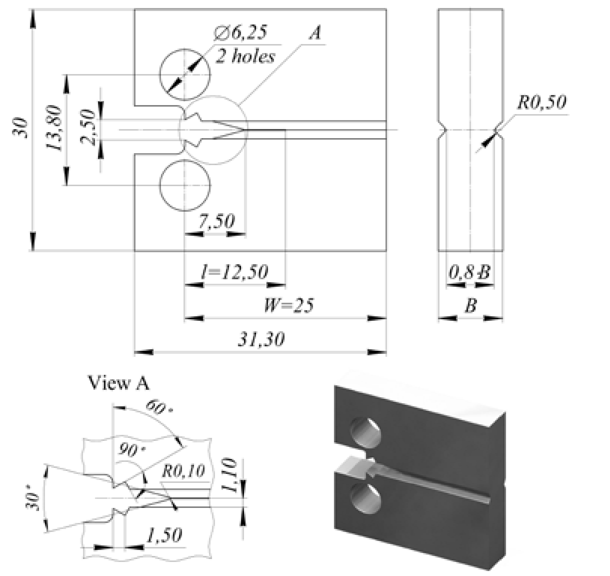
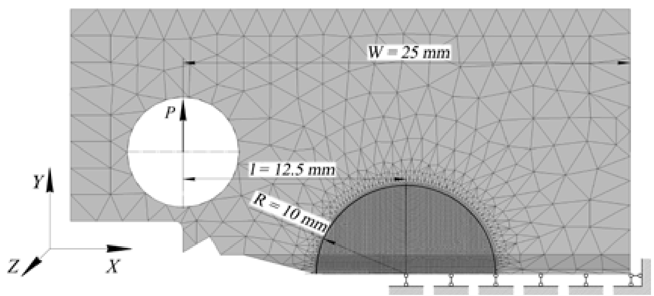
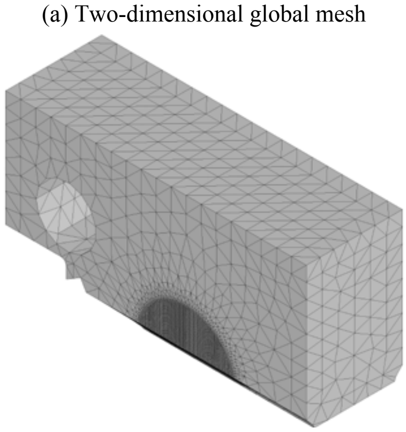


Figure 3. The geometrical model of the CT specimen.
Slika 3. Geometrijski model CT epruvete

The finite element model of the CT specimen is represented in Fig. 4. For the reason of minimization of the required computing resources, the finite element mesh is created in the one half of the geometric model (Fig. 4). For a more detailed account of the stress and strain distribution at the crack tip, the local mesh concentration with diameter of 10 mm is created around the crack tip.



(a) Two-dimensional global mesh



(b) Three-dimensional global mesh
Figure 4. Finite element model of the CT specimen.
Slika 4. Model konačnim elementima CT epruvete

A spatial finite element mesh is created in one of the modern CAE systems by means of decomposition of the internal volume of the geometrical model to the finite number of the small calculation elements having the spatial polygonal shape. Parabolic elements (second order elements) with tetrahedral shape and one intermediate node along each side are used in this numerical analysis. The use of such elements allows for achieving greater calculation accuracy due to the more accurate reproduction of the curvilinear surfaces of the geometric model, as well as a more accurate shape function which connects the displacement of the arbitrary point of the calculation element with the displacement of its nodes.

The creation of the load system is made in the modern CAE system. Fixed values of the vertical forces (force P) are added to the nodes situated on the cylindrical surface of the specimen hole. The value of K_{max} in Table 2 is obtained as the stress intensity factor corresponding to the maximum load P_{max} from the well-known equation in ASTM E399.

Table 2. Loading conditions of the CT specimen.
Tabela 2. Uslovi opterećenja CT epruvete

B/W	0.25	0.40	0.50
P_{max} (kN)	6.0	9.6	12.0
K_{max} (MPa·m$^{1/2}$)	66.0	66.0	66.0

Finite element calculations allow analyzing the size and the shape of the plastic zone in the vicinity of the crack tip by means of diagrams of von Mises equivalent stresses.

Algorithm for processing results of the numerical modelling

Sizes and shape of the plastic zone are analysed using the image data of the distribution of equivalent von Mises stresses in the vicinity of the crack tip.

Estimation of the influence of T-stress components on the plastic zone size is made in the mathematical package, where a special algorithm is created for processing the computed diagrams (Fig. 5). These diagrams must satisfy certain graphic conditions. Diagrams must be presented as monochromatic images of equivalent stress fields with clear gray gradation of stress, from white (stress level equal to yield stress σ_Y) to black (zero stress level). In accordance with the main idea of the algorithm, the special procedure realizes the consequent selection of pixels that belong to the image of the current diagram, along a radius from the crack tip with an appropriate step. If the colour of a current pixel differs from white, the procedure of pixel selection is stopped and current values of the radius and angle are saved. This radius corresponds to the boundary of the plastic zone in the vicinity of the crack tip.

DISCUSSION

In accordance with the above-mentioned methodology for theoretical estimation of the plastic zone size, the predicted results are compared with the results computed by FEM. Comparison of theoretical and FEM results of the plastic zone size estimation at the CT specimen thickness is presented in Fig. 6.

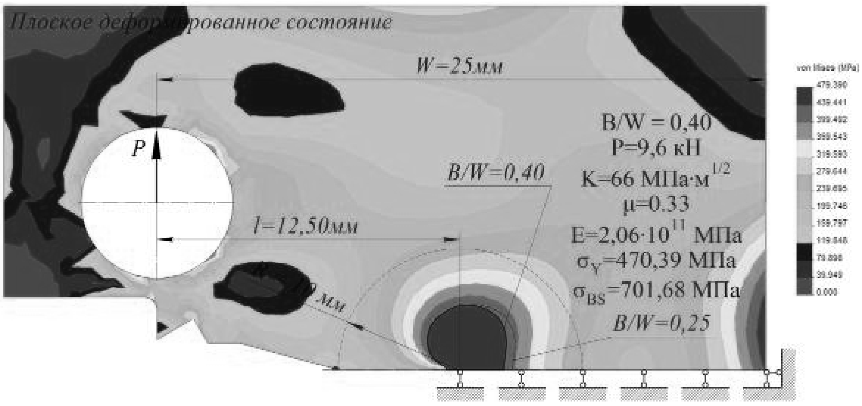


Figure 5. Typical image data of the distribution of equivalent von Mises stresses in the vicinity of the crack tip of the CT specimen.
Slika 5. Tipična slika raspodele ekvivalentnih fon Mizesovih napona u okolini vrha prsline CT epruvete

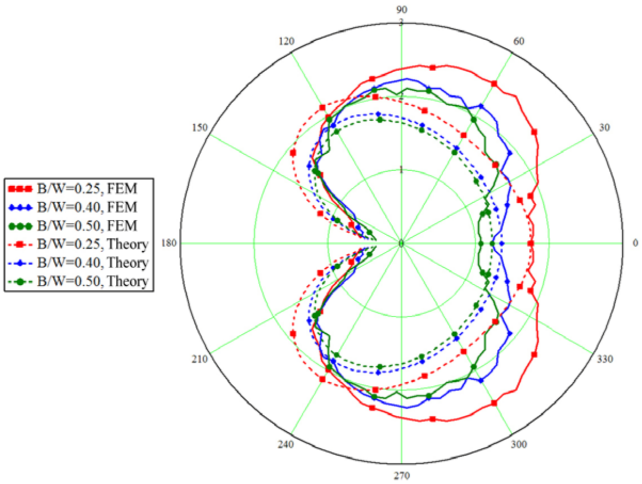


Figure 6. Comparison of theoretical and FEM results of the plastic zone estimation at the CT specimen thickness.
Slika 6. Poređenje teorijskih i FEM rezultata procene plastične zone po debljini CT epruvete

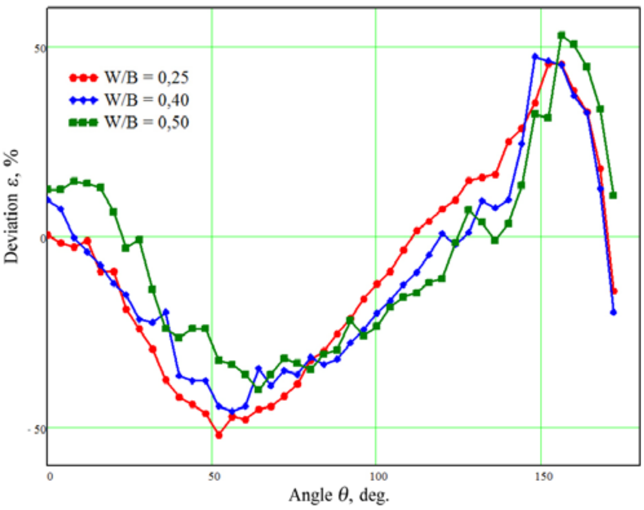


Figure 7. Deviation of theoretical plastic zone sizes from FEM results.
Slika 7. Odstupanje dimenzija teorijske plastične zone od FEM

It is observed that the deviation of theoretical plastic zone sizes from FEM results does not exceed 20% in angular intervals (0°, 30…45°) and (90°…100°, 135°… 145°) (Fig. 7). The deviation achieves a maximum and exceeds 40% in angular intervals from 135°…145° to 180° that can be connected with the particularity of the algorithm used during the processing of equivalent von Mises stress diagrams.

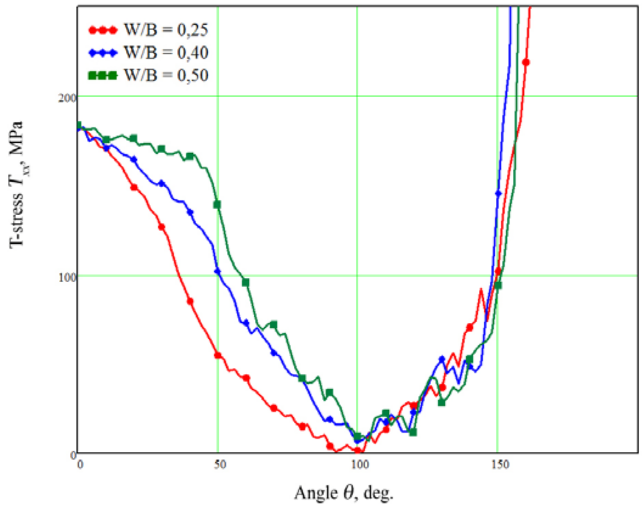


Figure 8. FE estimation of angular T_{xx}-stress distribution along the plastic deformation zone boundary at specimen thickness centre.
Slika 8. FE procena ugaone raspodele T_{xx}-napona duž granice zone plastične deformacije u sredini debljine epruvete

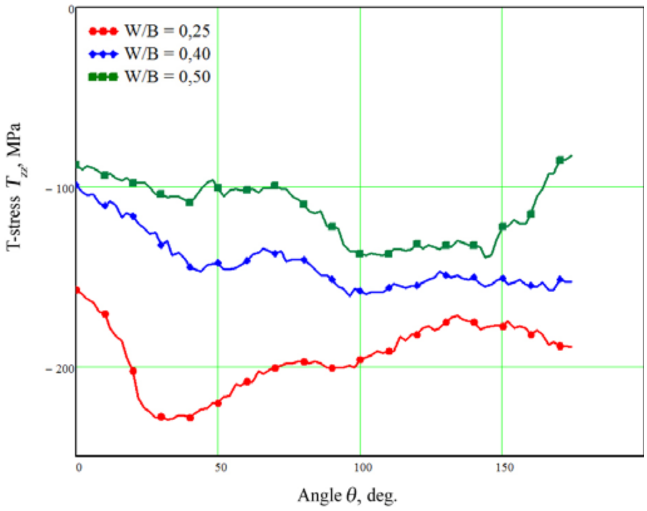


Figure 9. FE estimation of angular T_{zz}-stress distribution along the plastic deformation zone boundary at specimen thickness centre.
Slika 9. FE procena ugaone raspodele T_{zz}-napona duž granice zone plastične deformacije u sredini debljine epruvete

Probably, the high divergence between the results of the numerical, experimental, and analytical calculation at the interval (0°…145°) can be explained that actually T-stresses are not a constant and depend on angle θ. This assumption is corroborated by the results of FEM analysis (Figs. 8 and 9).

We can gather evidence that the dependence between T-stress components and the angular coordinate exists if we shall express K_I from Eq.(2) and substitute the received relation into Eq.(1). As a result we get the angular distribution of T_{xx}:

$$T_{xx} = \sigma_{xx} - \sigma_{yy}\left(\frac{3\cos\frac{\theta}{2} + \cos\frac{5\theta}{2}}{5\cos\frac{\theta}{2} - \cos\frac{5\theta}{2}}\right) \quad (23)$$

In a similar manner we receive the angular distribution of T_{zz}:

$$T_{zz} = \sigma_{zz} - \nu\left(\sigma_{yy}\left(\frac{8\cos\frac{\theta}{2}}{5\cos\frac{\theta}{2} - \cos\frac{5\theta}{2}}\right) + T_{xx}\right) \quad (24)$$

The received relations confirm earlier made assumptions about the necessity of introducing angular distributions of non-singular stress components into asymptotic formulas (1) and (3).

CONCLUSIONS

The theoretical analysis of the joint influence of non-singular T-stress components on the size of the plastic deformation zone at the tip of the mode I crack is carried out with attraction of asymptotic formulas, taking into account the triaxiality of the stress state at the tip of the mode I crack and von Mises yield criterion.

The size of the plastic zone at the central surface of the specimen decreases while increasing specimen thickness. It reflects the increase in the degree of deformation constraint at the crack tip by means of an increase in the non-singular stress T_{zz}.

Theoretical estimations of the plastic zone size with provision for T-stress components as a whole shows satisfactory results, especially in line of the crack continuation. This is especially important in the case of estimating reliable values of the fracture toughness. However, in some cases, the divergence between analytical calculation and FEM results exceeds 20%. For more correct determination of the shape of the plastic deformation zone it is necessary to take into account the angular distribution of non-singular T-stresses at the crack tip.

Certainly, accurate analysis and estimation of plastic zone sizes at mode I crack tip will promote the development of more correct criteria of validity in estimating the fracture toughness of engineering materials.

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