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THERMO ELASTIC-PLASTIC TRANSITION OF A HOMOGENEOUS THICK-WALLED CIRCULAR CYLINDER UNDER EXTERNAL PRESSURE

TERMO-ELASTO-PLASTIČNO PRELAZNO STANJE U HOMOGENOM DEBELOZIDNOM KRUŽNOM CILINDRU POD DEJSTVOM SPOLJAŠNJEG PRITISKA

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Abstract

Thermal elastic-plastic stresses have been obtained for thick-walled circular cylinder under external pressure by using the transition theory based on the concept of generalized principal strain measure that simplifies the constitutive equations by prescribing a priori the order of the measure of deformation, and that helps to achieve better agreement between theoretical and experimental results. Results have been analysed and discussed numerically as well as graphically. From our analysis, we can conclude that without the temperature gradient, the homogeneous circular cylinder with less compressibility is on the safer side of the design as compared to highly compressible circular cylinder, because the less compressible cylinder requires a high pressure for initial yielding as compared to the high compressible cylinder, while under effect of temperature, the highly compressible circular cylinder is on the safer side of the design as compared to less compressible circular cylinder, leading to the idea of 'stress saving' and minimizing the possibility of fracture.

Ključne reči
- toplotni uticaj
- elasto-plastično
- homogeno
- pritisak
- cilindar

Izvod

Određeni su termo-elasto-plastični naponi kod debelozidnog kružnog cilindra pod dejstvom spoljašnjeg pritiska, primenom teorije prelaznog naponskog stanja, zasnovanoj na konceptu mere generalisane glavne deformacije, kojom se pojednostavljuju konstitutivne jednačine kada se a priori dodeljuje red mere deformacije, i kojom se olakšava postizanje boljeg poklapanja teorijskih i eksperimentalnih podataka. Rezultati su analizirani i diskutovani kako numerički, tako i grafički. Iz naših analiza se zaključuje da kada nema temperaturskog gradijenta, homogeni kružni cilindar manje stišljivosti (veće krutosti) je pogodnija varijanta u projektovanju u odnosu na kružni cilindar veće stišljivosti (manje krutosti), jer je za cilindar manje stišljivosti potreban veći pritisak za pojavu tečenja u poređenju sa cilindrom veće stišljivosti. Kada se uvede temperaturski uticaj, kružni cilindar veće stišljivosti je tada pogodnija varijanta u projektovanju u poređenju sa kružnim cilindrom manje stišljivosti, što ide u prilog ideji o 'naponskoj uštedi' i smanjenju mogućnosti za pojavu loma.

INTRODUCTION

The constantly increasing industrial demand for axisymmetrical cylindrical and spherical components or their elements has concentrated the attention of designers and scientists on this particular area. The research on the prediction of stresses in a thick-walled hollow circular cylinder has never ceased because of the importance of these basic structures in numerous mechanical, civil, electrical and computer engineering applications. These days in the nuclear industry, cylinders subjected to external pressure have become a point of interest due to their application to advanced small and medium-sized light water reactors, for example, steam generator tubes and pipelines under seawater for transporting gas, oil, etc. Now, for a design and an integrity evaluation of a cylinder under external pressure, one should carefully consider the failure characteristics of a cylinder under internal and external pressure. The problems for a thick-walled cylinder under internal pressure for an isotropic material were discussed by many authors /1-4/. In analysing the problems, these authors used some simplifying assumptions, i.e. first, the deformation is small enough to make the infinitesimal strain theory applicable, second, simplifications are made regarding the constitutive equations of the material, like incompressibility of the material and an yield criterion. Incompressibility of the material is one of the most important assumptions that simplifies the problem. In fact, in most of the cases, it is not possible to find a solution in a closed form without this assumption. The transition theory /5-9/ does not require these assumptions and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. Gupta and Sharma /7/ solved the problem for an isotropic cylinder under internal pressure by using the transition theory.

In this paper, thermo elastic-plastic stresses for an isotropic thick-walled circular cylinder under external pressure are obtained by using Seth's transition theory /5/. Results obtained have been discussed numerically and depicted graphically.

Governing Equations

Consider a thick-walled circular cylinder of internal and external radii 'a' and 'b' respectively, subjected to external pressure p, temperature θ_0 applied at the internal surface.

The displacement components in cylindrical polar coordinates are given by /5-9/.

$$u = r(1-\beta),\ v = 0,\ \omega = dz \tag{1}$$

where β is a function of $r = \sqrt{x^2+y^2}$ and d is a constant.

The generalized principal strain measure /5/ is given by

$$e_{ii} = \int_0^{e_{ii}^A}\left[1-2e_{ii}^A\right]^{\frac{n}{2}-1} de_{ii}^A = \frac{1}{n}\left[1-\left(1-2e_{ii}^A\right)^{\frac{n}{2}}\right]$$

where 'n' is the measure.

From the above relation the generalized components of strain /5-9/ are given as following,

$$e_{rr} = \frac{1}{n}[1-(r\beta'+\beta)^n],\ e_{\theta\theta} = \frac{1}{n}[1-\beta^n],$$

$$e_{zz} = \frac{1}{n}[1-(1-d)^n],\ e_{r\theta} = e_{\theta z} = e_{rz} = 0 \tag{2}$$

The stress-strain relationships for an isotropic material is

$$T_{ij} = \lambda\delta_{ij}I_1 + 2\mu\varepsilon_{ij} - \xi\theta\delta_{ij} \tag{3}$$

where $I_1 = e_{kk}$; σ_{ij}, ε_{ij} are stress and strain tensors respectively, $\xi = \alpha(3\lambda + 2\mu)$, λ, μ are Lame's constants, δ_{ij} is the Kronecker delta, α is the coefficient of thermal expansion and θ is temperature.

Equation (3) can be written as

$$T_{rr} = \left(\frac{\lambda+2\mu}{n}\right)\left(1-(r\beta'+\beta)^n\right) + \frac{\lambda}{n}(1-\beta^n) + \lambda k - \xi\theta,$$

$$T_{\theta\theta} = \left(\frac{\lambda}{n}\right)\left(1-(r\beta'+\beta)^n\right) + \left(\frac{\lambda+2\mu}{n}\right)(1-\beta^n) + \lambda k - \xi\theta,$$

$$T_{zz} = \left(\frac{\lambda}{n}\right)\left(1-(r\beta'+\beta)^n\right) + \left(\frac{\lambda}{n}\right)(1-\beta^n) + (\lambda+2\mu)k - \xi\theta,$$

$$T_{zr} = T_{\theta z} = T_{r\theta} = 0 \tag{4}$$

where $k = \frac{1}{n}\left[1-(1-d)^n\right]$ and $\xi = \alpha(3\lambda+2\mu)$.

Equations of equilibrium are all satisfied except,

$$\frac{d}{dr}(T_{rr}) + \left(\frac{T_{rr}-T_{\theta\theta}}{r}\right) = 0 \tag{5}$$

The temperature field satisfying Fourier heat equation is

$$\nabla^2\theta = 0 \text{ and}$$

$$\theta = \theta_0 \text{ at } r = a,\ \theta = 0 \text{ at } r = b$$

where θ_0 is a constant, given by

$$\theta = \frac{\theta_0}{\log(a/b)}\log(r/b) \tag{6}$$

Substituting Eqs. (4) and (6) in Eq.(5), we get a nonlinear differential equation in β as,

$$\beta P(1+P)^{n-1}\frac{dP}{d\beta} + \left(P+\frac{C}{n}\right)(1+P)^n + \\ + P(1-C) - \frac{C}{n} + \frac{C\xi\bar{\theta}_0}{2\mu\beta^n} = 0 \tag{7}$$

where $r\beta' = \beta P$ and $\bar{\theta}_0 = \frac{\theta_0}{\log(a/b)}$.

The transitional points of P in Eq.(7) are $P \to 1$ and $P \to \pm\infty$. The boundary conditions are,

$$T_{rr} = 0 \text{ at } r = a;\ T_{rr} = -p \text{ at } r = b \tag{8}$$

The resultant force normal to plane z = const. must vanish,

$$2\pi\int_a^b rT_{zz}dr = p \tag{9}$$

Solution through the Principal Stresses

It has been shown that the asymptotic solution through the principal stress leads from elastic to plastic state at the transition point $P \to \pm\infty$, we define the transition function R, /5-9/, as,

$$R = T_{rr} - \frac{\lambda}{n}k + \alpha(3-2C)\theta = \\ = \frac{2\mu}{nC}\left[(2-C) - \beta^n(P+1)^n + (1-C)\right] + \\ + \lambda k\left(1-\frac{1}{n}\right) - \alpha(3-2C)\left(\frac{\lambda}{1-C}-1\right)\bar{\theta}_0\log\left(\frac{r}{b}\right) \tag{10}$$

where, $k = \frac{1}{n}\left[1-(1-d)^n\right]$.

Taking the logarithmic differentiation of Eq.(10) w.r.t. 'r', and integrating we get

$$R = Ar^{-C} \tag{11}$$

where A is a constant of integration and $C = \frac{2\mu}{\lambda+2\mu}$.

Using Eq.(11) in Eq.(10), we get

$$T_{rr} = Ar^{-C} + B - \alpha(3-2C)\theta \tag{12}$$

where $B = \frac{\lambda}{n}k$.

Using boundary conditions (8) in Eq.(12), we get

$$A = \frac{\alpha\theta_0(3-2C)+p}{a^{-C}-b^{-C}},\quad B = -p - Ab^{-C} \tag{13}$$

Substituting the value of A and B in Eq.(12), we get

$$T_{rr} = \frac{\alpha\theta_0(3-2C)+p}{a^{-C}-b^{-C}}\left(r^{-C}-b^{-C}\right) - p - \alpha\theta(3-2C) \tag{14}$$

Using the equation of equilibrium, we get

$$T_{\theta\theta} = \frac{\alpha\theta_0(3-2C)+p}{a^{-C}-b^{-C}}\left[(1-C)r^{-C}-b^{-C}\right] - \\ -\alpha\bar{\theta}_0(3-2C)\left(1+\log\left(\frac{r}{b}\right)\right) - p \tag{15}$$

The axial stress is obtained from Eq.(4) as

$$T_{zz}=\left(\frac{1-C}{2-C}\right)[T_{rr}+T_{\theta\theta}]+\left(\frac{\lambda C}{1-C}\right)\left(\frac{3-2C}{2-C}\right)e_{zz}-\frac{\lambda C\alpha\theta(3-2C)}{(1-C)(2-C)} \quad (16)$$

where,

$$e_{zz}=\frac{\frac{p}{2\pi}+\alpha\lambda\frac{C(3-2C)\theta}{(1-C)(2-C)}\left(\frac{b^2-a^2}{2}\right)-\left(\frac{1-C}{2-C}\right)\int_a^b(T_{rr}+T_{\theta\theta})rdr}{\frac{(3-2C)C}{(1-C)(2-C)}\left(\frac{b^2-a^2}{2}\right)} \quad (17)$$

From Eqs.(14) and (15), we get

$$T_{\theta\theta}-T_{rr}=-ACr^{-C}-\alpha\bar{\theta}_0(3-2C) \quad (18)$$

Initial Yielding

It is found that the value of $|T_{\theta\theta}-T_{rr}|$ is maximum at $r = a$, which means yielding of the cylinder will take place at the internal surface. Therefore, we have

$$|T_{\theta\theta}-T_{rr}|_{r=a}=\left|-ACa^{-C}-\alpha\bar{\theta}_0(3-2C)\right|=Y \quad (19)$$

The necessary pressure required for initial yielding is given by

$$|P_i|=\frac{p}{Y}=\left|\frac{1}{|A_2|}-\theta_1\left|\frac{A_3}{A_2}\right|\right| \quad (20)$$

where

$$A_3=-\left\{\frac{(3-2C)Ca^{-C}}{a^{-C}-b^{-C}}+\frac{3-2C}{\log\left(\frac{a}{b}\right)}\right\}$$

and $\theta_1=\frac{\alpha\theta_0}{Y}$.

In fully plastic state $C \to 0$, Eq.(18) becomes

$$T_{\theta\theta}-T_{rr}=\frac{-p-3\alpha\theta_0}{\log\left(\frac{b}{a}\right)}-3\alpha\bar{\theta} \quad (21)$$

At $r = b$

$$|T_{\theta\theta}-T_{rr}|_{r=b}=\left|\frac{-p-3\alpha\theta_0}{\log\left(\frac{b}{a}\right)}-\frac{3\alpha\theta_0}{\log\left(\frac{a}{b}\right)}\right|=Y_{10} \quad (22)$$

The pressure required for fully plastic state is given by

$$|P_f|=\frac{p}{Y_1}=\left|\frac{1}{A_4}\right| \quad (23)$$

where $A_4=\frac{1}{\log\left(\frac{b}{a}\right)}$.

The stresses for fully plastic state are obtained by taking $C \to 0$.

$$T_{rr}=\frac{(3\alpha\theta_0-p)\log\left(\frac{r}{b}\right)}{\log\left(\frac{b}{a}\right)}-p-3\alpha\theta \quad (24)$$

$$T_{\theta\theta}=\frac{(3\alpha\theta_0+p)\left(\log\left(\frac{r}{b}\right)-1\right)}{\log\left(\frac{b}{a}\right)}-p-3\alpha\theta-3\alpha\bar{\theta}_0 \quad (25)$$

$$T_{zz}=\frac{1}{2}(T_{rr}+T_{\theta\theta}) \quad (26)$$

In the non-dimensional state, we define

$$R_0=\frac{a}{b},\ R=\frac{r}{b},\ \theta_1=\frac{\alpha\theta_0}{Y},\ \sigma_r=\frac{T_{rr}}{Y},\ \sigma_\theta=\frac{T_{\theta\theta}}{Y},$$

$$\sigma_z=\frac{T_{zz}}{Y},\ P_i=\frac{p}{Y},\ P_f=\frac{p}{Y_1}.$$

The initial yielding and transitional stresses in non-dimensional form are given as

$$|P_i|=\frac{p}{Y}=\left|\frac{1}{|A_5|}+\theta_1\left|\frac{A_6}{A_5}\right|\right| \quad (27)$$

where $A_5=\frac{C}{1-R_0{}^C}$, and $A_6=-\left(\frac{(3-2C)C}{1-R_0{}^C}+\frac{3-2C}{\log(R_0)}\right)$.

The transitional stresses in non-dimensional form are given as,

$$\sigma_r=\frac{T_{rr}}{Y}=\frac{\theta_1(3-2C)+P_i}{R_0^C-1}(R^{-C}-1)-P_i-\frac{\theta_1(3-2C)\log(R)}{\log(R_0)} \quad (28)$$

$$\sigma_\theta=\frac{T_{\theta\theta}}{Y}=\frac{\theta_1(3-2C)+P_i}{R_0^C-1}((1-C)R^{-C}-1)-\frac{\theta_1(3-2C)(1+\log(R))}{\log(R_0)}-P_i \quad (29)$$

$$\sigma_z=\frac{T_{zz}}{Y}=\left(\frac{1-C}{2-C}\right)[\sigma_r+\sigma_\theta]+\left(\frac{\lambda C}{1-C}\right)\left(\frac{3-2C}{2-C}\right)e_{zz}+\frac{\lambda C\alpha\theta(3-2C)}{(1-C)(2-C)} \quad (30)$$

The pressure required for fully plastic state as $C \to 0$ in non-dimensional is given by

$$|P_f|=\frac{p}{Y_1}=\left|\log\left(\frac{1}{R_0}\right)\right| \quad (31)$$

The stresses for fully plastic state in non-dimensional are given as

$$\sigma_r=\frac{T_{rr}}{Y_1}=\frac{(-P_f-3\theta_1)\log(R)}{\log(R_0)}-P_f-\frac{3\theta_1\log(R)}{\log(R_0)} \quad (32)$$

$$\sigma_\theta=\frac{T_{\theta\theta}}{Y_1}=\frac{(-P_f-3\theta_1)(1+\log(R))}{\log(R_0)}-P_f-\frac{3\theta_1(1+\log(R))}{\log(R_0)} \quad (33)$$

$$\sigma_z=\frac{T_{zz}}{Y_1}=\frac{1}{2}(\sigma_r+\sigma_\theta) \quad (34)$$

NUMERICAL ILLUSTRATION AND DISCUSSION

To observe the combined effects of pressure and temperature on a cylinder made of homogeneous material, we draw the graph between pressure 'p' and radii ratio $R_0 = a/b$, at different temperatures $\theta_1 = 0$, 10 and 50.

For a homogeneous circular cylinder, yielding starts at internal surface. It is seen from Figure 1 that highly compressible cylinder requires less pressure to yield while less compressible homogeneous cylinder requires high pressure to yield. It has been observed that with the introduction of temperature, the pressure required for initial yielding goes on increasing. Also, with the increase in temperature, the required pressure increases very significantly. Also, it has been observed from Fig. 1 that with temperature gradient, the pressure required for initial yielding is high for highly compressible circular cylinder as compared to the less compressible circular cylinder.

From Figure 2, it has been observed that transitional stresses are maximum at internal surface. From Fig. 2, we can observe that without temperature, circumferential stresses are high for the highly compressible cylinder as compared to the less compressible cylinder. Also, it has been observed that as pressure increases, circumferential stresses increases significantly. From Figs. 3-5, it has been observed that with the introduction of temperature, the circumferential stress increases. With the increase in pressure, again circumferential stresses increases significantly.

From Figure 6, we can see that fully plastic stresses are of compressible nature and are maximum at the internal surface. From Figures 7-9, it can be seen that with the increase in pressure and temperature, compressible circumferential stresses increases significantly.

CONCLUSION

From the above analysis, we can conclude that without the temperature gradient, the less compressible circular cylinder is on the safer side of the design as compared to the highly compressible circular cylinder, while with the temperature gradient, the highly compressible circular cylinder is on the safer side of the design as compared to less compressible circular cylinder. This is because of the reason that pressure required for initial yielding is high for the highly compressible circular cylinder as compared to the less compressible circular cylinder.

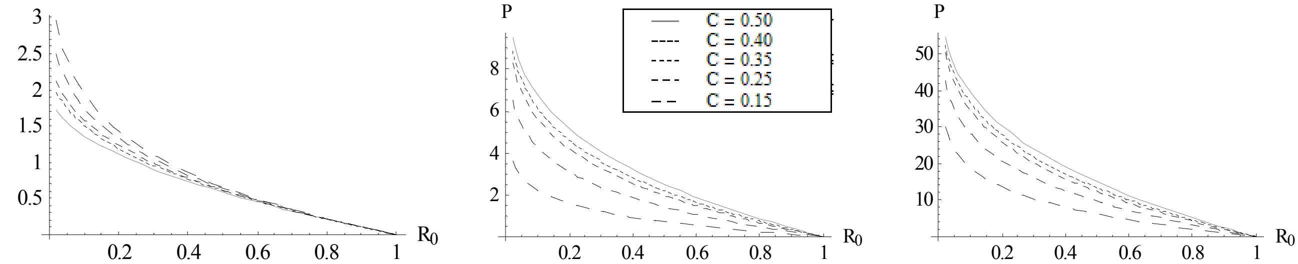

Figure 1. External pressure required for initial yielding (at different temperature i. e. $\theta_1 = 0, 10, 50$) of a homogeneous circular cylinder.
Slika 1. Spoljni pritisak potreban za inicijaciju tečenja (na različitim temperaturama, na pr. $\theta_1 = 0, 10, 50$) homogenog kružnog cilindra

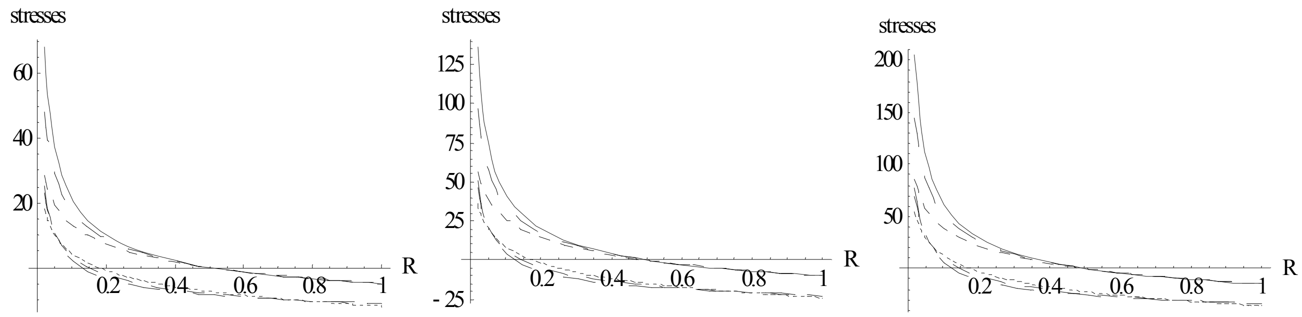

Figure 2. Transitional stresses for a thick-walled cylinder under different external pressure at 5, 10 and 15 respectively.
Slika 2. Prelazni naponi kod debelozidnog cilindra pod različitim spoljnim pritiscima 5, 10 i 15, respektivno

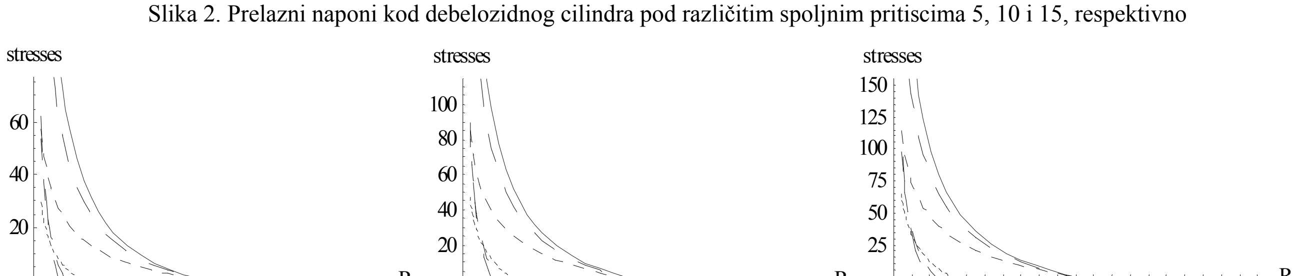

Figure 3. Thermal ($\theta_1 = 10$) transitional stresses for a thick-walled cylinder under different external pressure at 5, 10 and 15, respectively.
Slika 3. Termički ($\theta_1 = 10$) prelazni naponi kod debelozidnog cilindra pod različitim spoljnim pritiscima za 5, 10 i 15, respektivno

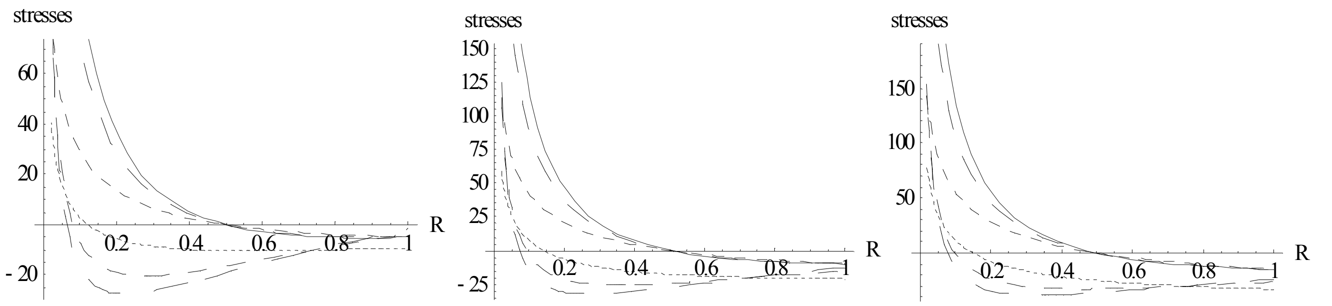

Figure 4. Thermal (θ_1 = 20) transitional stresses for a thick-walled cylinder under different external pressure at 5, 10 and 15, respectively.
Slika 4. Termički (θ_1 = 20) prelazni naponi kod debelozidnog cilindra pod različitim spoljnim pritiscima za 5, 10 i 15, respektivno

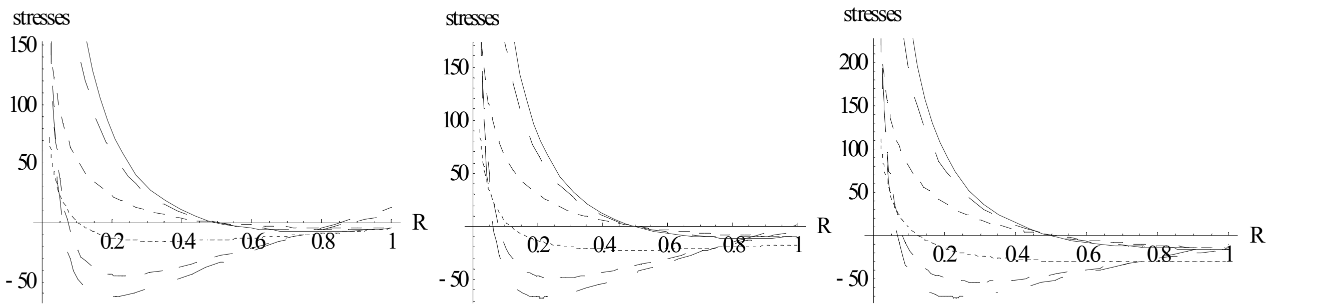

Figure 5. Thermal (θ_1 = 50) transitional stresses for a thick-walled cylinder under different external pressure at 5, 10 and 15, respectively.
Slika 5. Termički (θ_1 = 50) prelazni naponi kod debelozidnog cilindra pod različitim spoljnim pritiscima za 5, 10 i 15, respektivno

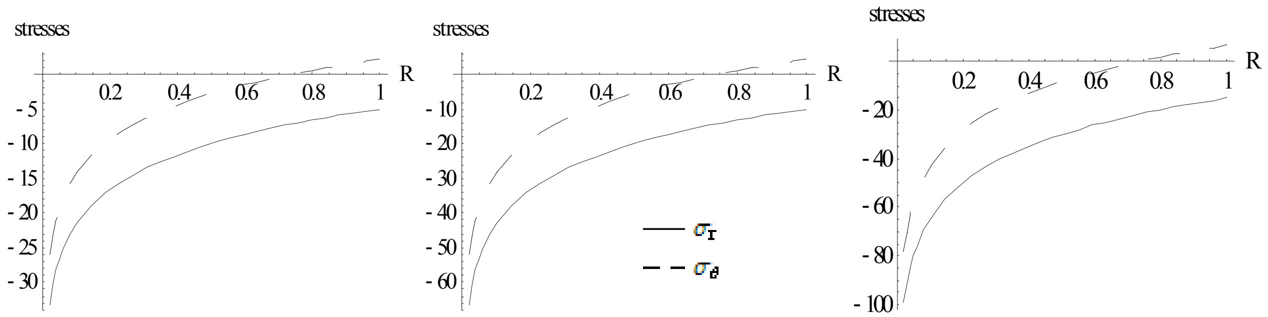

Figure 6. Fully plastic stresses for a thick-walled cylinder under different external pressure at 5, 10 and 15, respectively.
Slika 6. Potpuni plastični naponi kod debelozidnog cilindra pod različitim spoljnim pritiscima, 5, 10 i 15, respektivno

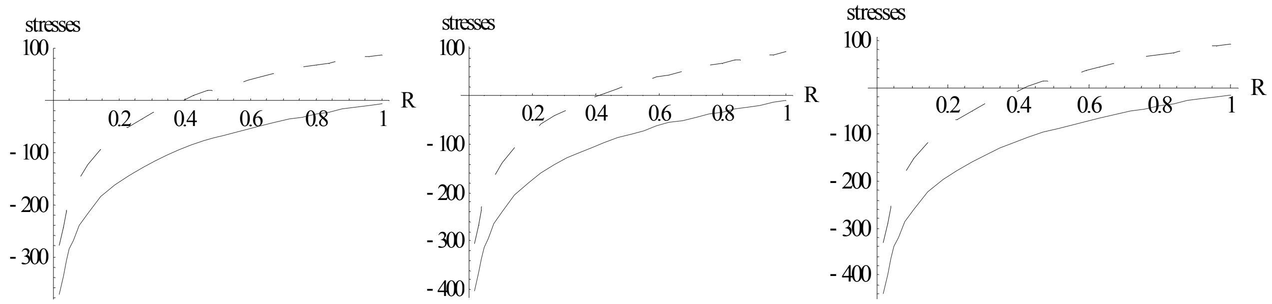

Figure 7. Thermal (θ_1 = 10) fully plastic stresses for a thick-walled cylinder under different external pressure at 5, 10 and 15, respectively.
Slika 7. Termički (θ_1 = 10) potpuni plastični naponi kod debelozidnog cilindra pod različitim spoljnim pritiscima za 5, 10 i 15, respektivno

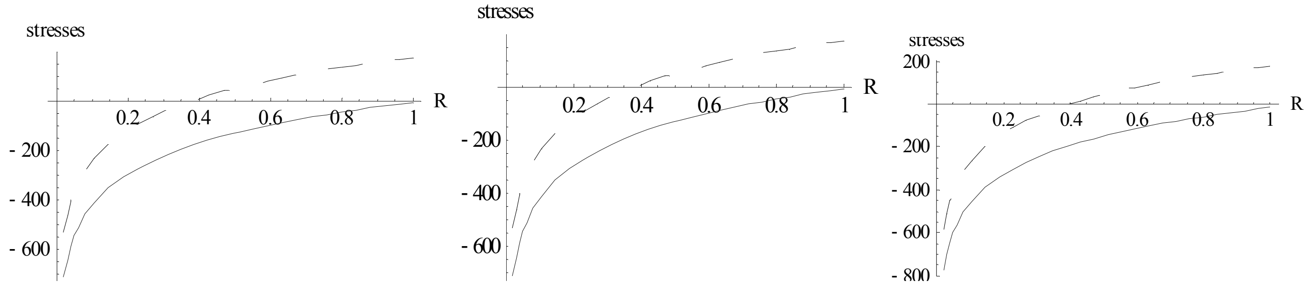

Figure 8. Thermal (θ_1 = 20) fully plastic stresses for a thick-walled cylinder under different external pressure at 5, 10 and 15, respectively.
Slika 8. Termički (θ_1 = 20) potpuni plastični naponi kod debelozidnog cilindra pod različitim spoljnim pritiscima za 5, 10 i 15, respektivno

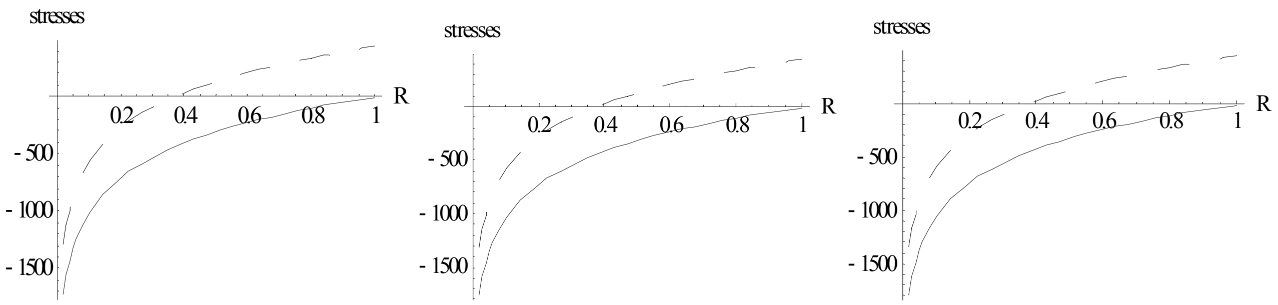

Figure 9. Thermal (θ_1 = 50) fully plastic stresses for a thick-walled cylinder under different external pressure at 5, 10 and 15, respectively.
Slika 9. Termički (θ_1 = 50) potpuni plastični naponi kod debelozidnog cilindra pod različitim spoljnim pritiscima za 5, 10 i 15, respektivno

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