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COMPARISON BETWEEN NUMERICAL METHODS FOR HASOFER-LIND'S RELIABILITY INDEX COMPUTATIONS

POREĐENJE NUMERIČKIH METODA ZA POUZDANOST PRORAČUNSKOG POKAZATELJA HASOFER-LINDA

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Abstract

Since several years, the trends to designing structures are towards probabilistic design more than deterministic design. This is based on the fact that a null risk does not exist. Engineers now prefer to speak in terms of probability of failure. This means that a given risk of failure is accepted. In counterpart, a cheaper structure is expected.

The problem is in determining the pertinent failure criterion and the corresponding random input variables. If the laws of distribution of mechanical properties are generally well known, it is more difficult to have those representing the type and size of defects, or loading. Some experiments should be done to obtain the desired data.

There are several reliability indexes that are presented in this work. We will focus with the following on the Hasofer and Lind's index, that exhibits some theoretical advantages in conjunction with easiness of computation.

Four methods of computation are presented: a "black box", associated with finite elements software, a numerical method implemented with the help of symbolic computation software, the same but operating with a spreadsheet and the well-known Monte-Carlo method. A calibration test and an example have been treated with these methods and the results are compared in terms of precision, speed, easiness of programming and cost.

Ključne reči
- konstrukcije
- pouzdanost
- pokazatelj pouzdanosti
- numeričke metode
- verovatnoća otkaza
- Hasofer-Lindov pokazatelj

Izvod

Već više godina trend u projektovanju konstrukcija je više usmeren na probabilističke nego na determinističke metode. Razlog je činjenica da rizik uvek postoji. Zbog toga inženjeri danas rađe govore o verovatnoći loma. To znači da je prihvaćeno postojanje određenog rizika od loma. U tom smislu očekuje se jeftinija konstrukcija.

Problem je da se odredi pogodan kriterijum otkaza i odgovarajući unos slučajnih podataka. Ako je zakonitost raspodele mehaničkih osobina načelno dobro poznata, mnogo je teže vladati onim zakonima koji predstavljaju vrstu greške, veličinu greške, ili opterećenje. Da bi se dobili željeni podaci, potrebno je izvesti određene eksperimente.

U ovom radu su prikazani neki pokazatelji pouzdanosti. U daljem tekstu će pažnja biti usmerena na Hasofer i Lindov pokazatelj, koji ima izvesne teorijske prednosti a povezane su sa pogodnošću proračuna.

Prikazane su četiri metode proračuna: „crna kutija", koja je povezana sa softverom konačnih elemenata, jedna numerička metoda koja je uvedena pomoću proračunskog softvera simbolima, ista takva koja koristi tabelu podataka i poznatu metodu Monte-Karlo. Test kalibracije i jedan primer su razmotreni pomoću ovih metoda, a rezultati su upoređeni u pogledu preciznosti, brzine, jednostavnosti programiranja i cene.

INTRODUCTION

Since several years, design codes are changing from deterministic to probabilistic design. This trend is clearly illustrated in Fig. 1, /1/. It shows that, when knowing a function of variables, which is defined as a failure criterion, in the space of these variables this function defines two zones: the "safe zone" and the "failure zone". If variables have determined values, the deterministic approach clearly separates the non-dangerous variable combinations from dangerous ones. In general, variables are mainly mechanical characteristics, which are subjected to some variability. Consequently, the deterministic approach is no more valid, and a probabilistic approach should be implemented. In Fig. 1, a third "security zone" is clearly seen, where the probability of failure is practically zero. Between the failure zone and security zone, in the safe zone, there exists some probability of failure, which should be computed.

When defining the failure criterion function, there are two kinds of variables: fixed variables with a unique value, and random variables, the values of which are depending on a particular probability density.

Two main processes can be examined. One is based on the well-known Monte-Carlo procedure, the other on the computation of an index of reliability. Transformation from the index of reliability to probability of failure is possible as these two quantities are related together by a Normal Distribution probability function.

Both procedures and the numerical processes associated with are presented in this paper. Differences are found between them and tentative explanations are proposed. It is therefore necessary to determine a procedure to estimate the probability of failure.

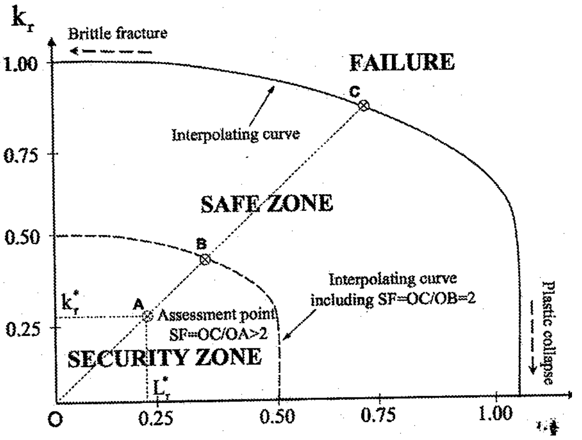

Figure 1. Failure Assessment Diagram with three zones, /1/.
Slika 1. Dijagram analize loma sa tri zone, /1/

THE MONTE-CARLO PROCEDURE

The first paper speaking about Monte-Carlo procedure was published by Metropolis and Ulam, /2/, in 1949. In its principle, it is quite simple: for a given problem implying random variables, with a binary solution (e.g. True or False), this result is computed for a large set of random variables and the probability of success (here True, for instance), is the number of sets giving True over the total number of sets tested.

For instance, we consider the problem of a quantity computed from random variables as input data and which should not exceed a critical value. To obtain the probability of fracture, it is just necessary to generate a large amount of sets of input variables and then to compute and test the criterion, the ratio of sets leading to fracture over the total amount of sets being this probability of fracture.

However, two main limits with this technique are:

– If there is some natural correlation between two variables, or some physical constraint, only sets complying with these conditions should be considered. This could imply some programming difficulties and reduce the total amount of data sets actually used.
– If the expected probability is very low (for instance, less than 10^{-7}, in order to obtain relatively stable results it is necessary to compute the ratio with, at least, more than 10^{8} valid sets of data. Moreover, with such low probability, it is necessary to test the variability and scatter of the result.

These computations are not very difficult, but may be extremely computer time consuming.

THE RELIABILITY INDEX METHODS

There are several reliability indexes that are defined. It is necessary first to recall what a reliability index is.

The state of a structure can be defined from a set of variables which define a criterion. For instance, if the strength S and the stress Σ in a point of a structure are given, the equation:

$$S = \Sigma \tag{1}$$

is the criterion. This can be written as:

$$M = S - \Sigma = 0 \tag{2}$$

and if the left side is negative, there is fracture. Giving randomly distributed S and Σ strength and stress, there are sets that give fracture and others do not. Equation (2) is called "state function" and we will focus on the probability that this function is positive, i.e. the structure is safe. From the statistical distribution, values of M quantities called "reliability indexes" are derived.

Cornell's reliability index

For each set of variables, the state function M is computed and, therefore, an expected value (mean value) $E(M)$ and a standard deviation $V(M)^{0.5}$ are computed. When the distribution of each random variable is normal, the distribution of the state function M is expected to be also a standard normal one. Cornell's reliability index is expressed in terms of the ratio:

$$\beta_c = \frac{E(M)}{V(M)^{1/2}} \tag{3}$$

This index is presented in Fig. 2: g^0 is $E(M)$ and σ_g is $V(M)^{1/2}$. Probability of fracture P_f is given in the figure.

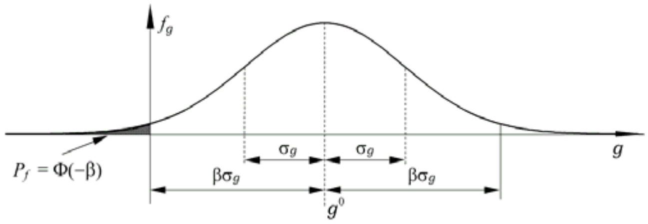

Figure 2. Representation of the β_c reliability index, /3/.
Slika 2. Prikaz β_c indeksa pouzdanosti, /3/

Hasofer-Lind's reliability index, /4/

But, Cornell's reliability index presents a lack of invariance with respect to the formulation of the state function. This can be solved by computing the index at a "design point" instead of the mean values that were used in Cornell's index. This "design point" should be determined by an iteration technique, working in the space of transformed standard random variables.

$$z_i = \frac{x_i - \overline{x_i}}{\sigma_{x_i}} \tag{4}$$

Hasofer-Lind's reliability index is therefore defined as the shortest distance between the origin of reduced variable space and the limit state function, as shown in Fig. 3.

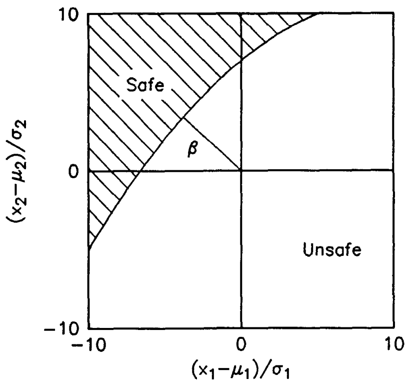

Figure 3. Definition of Hasofer-Lind's index in bi-dimensional space, /4/.
Slika 3. Definicija Hasofer-Lindovog pokazatelja u dvodimenzionalnom prostoru, /4/

The mathematical process to compute this index consists in an optimization procedure to determine the minimum distance between origin and state function. Constraints in this procedure are such that data sets should verify the limit state function. Moreover, in order to avoid problems due to the eventual existence of several local minimums, it is recommended to define a domain of validity for each variable. But this does not completely avoid the problem.

Of course, in the example given in Fig. 4, there are only two variables, and the distance from the origin of reduced variables to the limit state function (a curve) is clearly visible. In a general case of *n* variables, the limit state function is a surface in the *n-space* and Hasofer-Lind's index is the distance from origin to that surface.

As the β_c Cornell's index, the β_{HL} Hasofer-Lind's index is related to the probability of failure by a relationship:

$$P_f = 1 - \Phi\left(\beta_{HL}\right) \qquad (5)$$

where Φ is a cumulative density function (CDF) of the normal standard distribution. This distribution can be used, as the random variables have been normalized. Extensive details on these computations can be found in /5/.

COMPUTATION TOOLS

As mentioned before, two kinds of computations have been done:
- Monte-Carlo method, i.e. direct computation of the probability of failure;
- Hasofer-Lind's index, using a FORM (First Order Reliability Method).

In both cases, a comparison will be established using either Hasofer-Lind's index, or the probability of failure.

For the Monte-Carlo method, we have used the Mathematica® software published by Wolfram Company. For the Hasofer-Lind's index computation, we used three different tools:
- A built-in tool from the Cast3M® finite elements programme published by the French Atomic Energy Agency (CEA), also uses a built-in optimization procedure.
- Mathematica® software, using a built-in minimization function (FindMinimum).
- Excel® software, by Microsoft Company, using the built-in "Solver" function.

Validation procedure

For both methods and for all tools, it has been necessary to write a specific programme or process, and a validation of these programmes is needed. To achieve this, an example given by the Castem documentation has been used. It is a simple problem of tensile fracture of a homogeneous cylinder. The data are the net section of the cylinder, the applied load, and strength.

The limit function, in this simple case is:

$$A \times \sigma_u - P = 0 \qquad (6)$$

where A is the net section, σ_u the strength, and P the applied load.

Random variables are the applied load and strength. Data are summarized in Table 1.

Table 1. Validation data for a cylinder submitted to tension.
Tabela 1. Podaci za ocenu cilindra izloženog zatezanju

	Net section	Load	Strength
	m^2	MN	MPa
Mean/given value	0.42	70	272.72
Standard deviation	–	15	16.36

Results of performed computations are given in Table 2.

It can be seen that all methods approximately give quite the same result. This validates the four methods and allows us to use them for more sophisticated computations. It should be noticed that the Monte-Carlo, Mathematica and Excel methods all run on the basis of the generation of several sets of random data. Consequently, from one computation to another, the result is slightly different, and results given here are actually mean values of five computations. For each method, individual results are very close together. The mean value, standard deviation and standard error are given in Table 3. Although the errors remain low, it should be noticed that the probability of failure is about 10 times higher as that of Hasofer-Lind's index.

Table 2. Comparison between the different computation methods for reliability assessment.
Tabela 2. Poređenje različitih računskih metoda za ocenu pouzdanosti

	Monte-Carlo	Castem	Mathematica	Excel
Probability of failure, P_f	$3.414\cdot10^{-3}$	$3.470\cdot10^{-3}$	$3.369\cdot10^{-3}$	$3.489\cdot10^{-3}$
Hasofer-Lind's index, HL	2.7051	2.6997	2.6962	2.6980

Table 3. Comparison of results for the tensile test.
Tabela 3. Poređenje rezultata ispitivanja zatezanjem

	Mean	Standard deviation	Standard error
Probability of failure, P_f	$3.436\cdot10^{-3}$	$5.45802\cdot10^{-5}$	1.6%
Hasofer-Lind's reliability index	2.6998	0.0038	0.1%

APPLICATION TO A CRACKED TUBE

These methods have been applied to the problem of a cylindrical tube under internal pressure and exhibiting a semi-circular external and axial crack. The crack in the wall of the pipe is presented in Fig. 4. Tube dimensions are: inner diameter 219.1 mm, wall thickness 6.1 mm.

Other data are given in Table 4.

To perform the computation, the internal pressure is needed. Four values are chosen: 21.5, 23, 24 and 25 MPa.

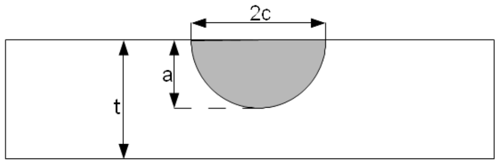

Figure 4. Definition of an axial semi-circular crack in the wall of a cylindrical pipe.
Slika 4. Definicija aksijalne polukružne prsline u zidu cilindrične cevi

Table 4. Input data for random variables. All distributions are Gaussian.
Tabela 4. Ulazni podaci za slučajne promenljive. Sve raspodele su Gausove

Data for Standard Normal Distribution	*Crack depth*	*Yield stress*	*Ultimate tensile strength*	*Plane strain fracture toughness*
	a, mm	σ_y, MPa	σ_u, MPa	K_{Ic}, MNm$^{-3/2}$
Mean value	3.0	410.0	528.0	121.0
Standard deviation	0.3	41.0	52.8	12.4

Limit function

Fracture analysis of the cylinder is achieved with SINTAP proposed by Gubeljak, /6, 7/. The relationships for the default level are used:

$$f\left(L_r\right)=\left[1+\frac{1}{2}L_r^2\right]^{-1/2}\times\left[0.3+0.7\exp\left(-0.6L_r^6\right)\right]$$

for $1\leq L_r\leq L_r^{\max}$ with $L_r^{\max}=1+\left[\dfrac{150}{\sigma_y}\right]^{2.5}$ and $L_r=\dfrac{\sigma_g}{\sigma_f}$

where σ_g is the gross stress if $L_r\leq L_r^{\max}$

$L_r=L_r^{\max}$ if $\dfrac{\sigma_g}{\sigma_f}\geq L_r^{\max}$ and $\sigma_f=\dfrac{\sigma_y+\sigma_u}{2}$ is the flow stress, σ_u being the ultimate stress.

f(L) represents the limit function, and is equal to:

$$f\left(L_r\right)=k_r=\frac{K_{app}}{K_{Ic}} \tag{8}$$

K_{app} is the applied stress intensity factor, and K_{1c} is the critical stress intensity factor (plane strain fracture toughness).

For an axial external semi-circular crack, the maximum applied stress intensity factor is given by /8/:

$$K_{app}=\sigma_g\sqrt{\pi a}F\left(\frac{R_i}{t},\frac{2c}{a},\frac{a}{t}\right) \tag{9}$$

where $F\left(\dfrac{R_i}{t},\dfrac{2c}{a},\dfrac{a}{t}\right)$ is a shape factor, including the curvature effect. In our case, this factor is estimated to 0.675.

The resulting failure assessment diagram is presented in Fig. 5. The limit function is therefore:

$$f\left(L_r\right)-k_r=0$$

$$\left[1+\frac{1}{2}L_r^2\right]^{-1/2}\times\left[0.3+0.7\exp\left(-0.6L_r^6\right)\right]-\frac{\sigma_g\sqrt{\pi a}F\left(\frac{R_i}{t},\frac{2c}{a},\frac{a}{t}\right)}{K_{Ic}}=0 \tag{10}$$

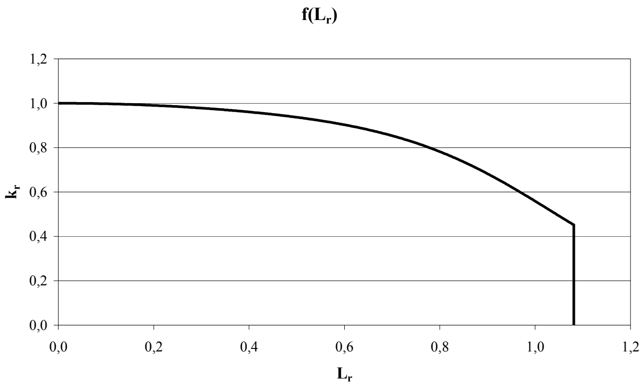

Figure 5. Presentation of the $f(L_r)$ function.
Slika 5. Prikaz funkcije $f(L_r)$

Results

As indicated before, results are given both in terms of Hasofer-Lind's index and probability of failure. They are presented in Fig. 6. The results are also summarized in Table 5Table . Finally, the results are also analysed, for each hoop stress, in terms of mean, standard deviation and standard error (Table 6).

It can be seen that the standard error for both HLI and P_f decreases with increasing hoop stress. But, if this error is reasonable for HLI, it is quite important in terms of P_f. In the case of the first line, if we do not take into account the result from the Monte-Carlo method, we obtain the results given in Table 7.

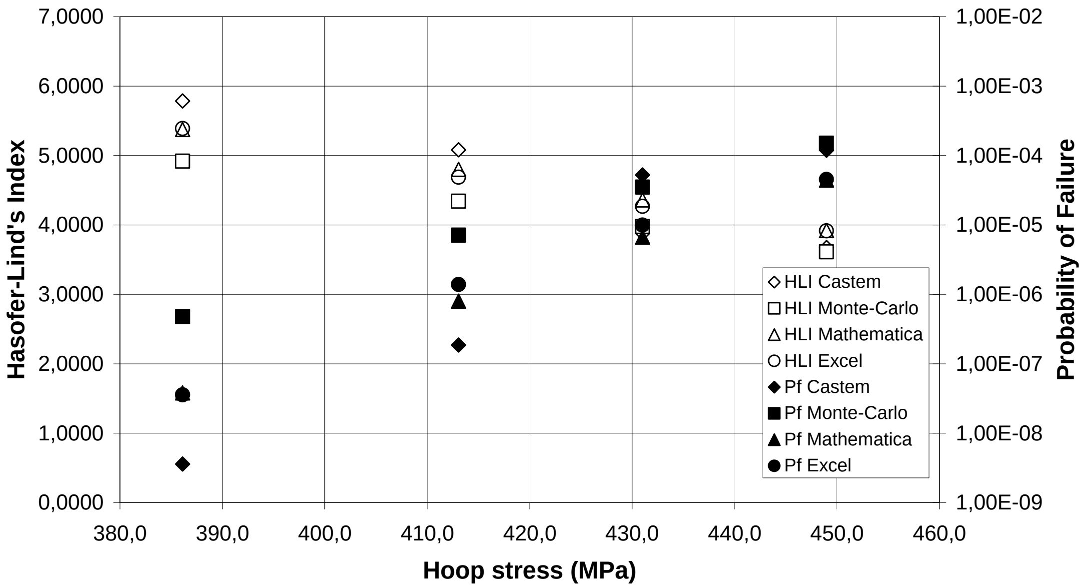

Figure 6. Hasofer-Lind's Index and probability of failure for different methods of computation.
Figure 6. Hasofer-Lindov pokazatelj i verovatnoća otkaza za različite metode proračuna

Table 5. Results of computations in terms of Hasofer-Lind's index (HLI) and probability of failure (P_f).
Tabela 5. Rezultati proračuna u zavisnosti od Hasofer-Lindovog pokazatelja (HLI) i verovatnoća loma (P_f)

	Monte-Carlo		*Castem*		*Mathematica*		*Excel*	
Hoop stress, MPa	HLI	P_f	HLI	P_f	HLI	P_f	HLI	P_f
386.1	4.9180	4.75E-07	5.7863	3.60E-09	5.3769	3.79E-08	5.3879	3.57E-08
413.1	4.3408	7.13E-06	5.0826	1.86E-07	4.8008	7.97E-07	4.6872	1.39E-06
431.1	3.9757	3.51E-05	3.8791	5.24E-05	4.3591	6.70E-06	4.2670	9.92E-06
449.0	3.6137	1.50E-04	3.6736	1.20E-04	3.9202	4.42E-05	3.9152	4.52E-05

Table 6. Mean value, standard deviation (SD) and standard error (SE) between the different methods for each applied hoop stress.
Tabela 6. Srednja vrednost, standardna devijacija (SD) i standardna greška (SE) različitih metoda za svaki primenjeni obimski napon

Hoop stress MPa	*HLI*			P_f		
	mean	SD	SE	mean	SD	SE
386.1	4.2938	0.3549	8.27%	1.38E-07	2.05E-07	148%
413.1	3.7823	0.3069	8.12%	2.38E-06	2.97E-06	125%
431.1	3.2962	0.2292	6.95%	2.60E-05	2.21E-05	85%
449.0	3.0245	0.1601	5.29%	8.99E-05	6.14E-05	68%

Table 7. Mean value, standard deviation (SD) and standard error (SE) for the first case without Monte-Carlo computations.
Tabela 7. Srednja vrednost, standardna devijacija (SD) i standardna greška (SE) za prva tri slučaja bez Monte-Karlo proračuna

Hoop stress MPa	*HLI*			P_f		
	Mean	SD	SE	Mean	SD	SE
366.1	5.5170	0.2333	4.23%	2.57E-08	1.92E-08	75%

It is obvious that scattering of results is decreased. This is explained by the fact that our computer configuration allows us to make computations only for 10^7 sets of data, and the result is probably erroneous for the Monte-Carlo method at such low probability of failure.

However, it should be observed that the obtained probability of failure is, in any case, subjected to high variability.

CONCLUSION

It has been shown that the probability of failure of a defective structure can be evaluated using several different methods, either directly with a Monte-Carlo method, or indirectly with the help of Hasofer-Lind's index. A simple testing case gives quite good and homogeneous results. However, application to a cracked piped shows large scattering of results, particularly for the probability of failure. This is probably due to the fact that computed probability of failure is rather low, comparing to the calibration example. However, the Monte-Carlo method, Mathematica and Excel computations are very simple routines to write and to use. The Castem routine needs special knowledge of the Castem software.

For practical use, it is recommended to be very careful when computing in the domain of very low probability of failure.

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